Conformal Geometry in the Bulk
A Boundary Calculus for Conformally Compact Manifolds

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arXiv:1104.2991
with Rod Gover

arXiv:1104.4994, 1007.1724, 1003.3855, 0911.2477, 0903.1394, 0812.3364, 0810.2867
with Roberto Bonezzi, Olindo Corradini, Maxim Grigoriev, Emanuele Latini and Abrar Shaukat
Main Idea


*arXiv:math/0412393*
Describing the Boundary

\[ M = \text{bulk} \]

\[ (M, [g]) = \text{conformal manifold} \]

\[ g_{\mu\nu} \sim \Omega^2(x)g_{\mu\nu} \]

\[ \Sigma = \text{boundary} \]

inherits conformal class of metrics

\[ (\Sigma, [g_\Sigma]) \]

To say *where* the boundary is introduce an *almost* everywhere positive function \( \sigma(x) \).

\( \Sigma \) is the zero locus of \( \sigma \).
The Scale

Along $\Sigma$, the function $\sigma$ encodes boundary data, in the bulk it is a spacetime varying Planck Mass/ Newton Constant.

- The *scale* $\sigma(x)$ is the gauge field for local choices of units
  \[ \sigma(x) \sim \Omega(x)\sigma(x) \]

- The double equivalence class $[g_{\mu\nu}, \sigma] = [\Omega^2 g_{\mu\nu}, \Omega\sigma]$ determines a canonical metric $g^0$ AWAY FROM $\Sigma$
  \[ [g_{\mu\nu}, \sigma] = [g^0_{\mu\nu}, 1] \]
  in units $\kappa = 1$.

- **Example:** AdS
  \[ ds^2_0 = \frac{dx^2 + h(x)}{x^2}, \quad \sigma_0 = 1, \]
  but along $\Sigma$ should use
  \[ ds^2 = dx^2 + h(x), \quad \sigma = x, \]
  well defined at the boundary $\Sigma = \{x = 0\}$.
The Normal Tractor

Along $\Sigma$, $\nabla \sigma$ encodes the normal vector $n$ and $\nabla \cdot n$ the mean curvature $H$

Introduce *scale* and *normal* tractors

$$I = \begin{pmatrix} \sigma \\ \nabla \sigma \\ -\frac{1}{d}(\Delta \sigma + \sigma J) \end{pmatrix}, \quad N = \begin{pmatrix} 0 \\ \hat{n} \\ -H \end{pmatrix},$$

$d := \dim M$, $R_{\mu\nu\rho\sigma} := W_{\mu\nu\rho\sigma} + (g_{\mu\rho}P_{\nu\sigma} \pm 3 \text{ more})$, $J := P^\mu_\mu$

**Theorem (Gover)**

$$I^2 = 1 \implies I|_{\Sigma} = N$$

and $(M, [g], \sigma)$ asymptotically hyperbolic.

Away from $\Sigma$ the scale tractor controls bulk geometry, along $\Sigma$ it carries boundary information.
Tractors

Probably not surprising that to describe conformal geometry, one should use 6-vectors rather than 4-vectors! Spacetime remains 4-dimensional.

Weight $w$ tractors are defined by their gauge transformation w.r.t. conformal transformations

$$T^M := \begin{pmatrix} T^+ \\ T^m \\ T^- \end{pmatrix} \mapsto \Omega^w \begin{pmatrix} \Omega & 0 & 0 \\ \gamma^m & \delta_n^m & 0 \\ -\frac{1}{2\Omega} \gamma^2 & -\frac{1}{\Omega} \gamma_n & \frac{1}{\Omega} \end{pmatrix} \begin{pmatrix} T^+ \\ T^n \\ T^- \end{pmatrix} =: \Omega^w U^M_N T^N,$$

Here $\gamma_\mu := \Omega^{-1} \partial_\mu \Omega$ and $U^M_N \in SO(d, 2)$.

Tractors give a tensor calculus for conformal geometry ♦.

Example: $T^2 := 2 T^+ T^- + T^m T_m$ is a conformal invariant

Parallel Scale Tractors

The tractor bundle is physically very interesting because parallel tractors correspond precisely to Einstein metrics.

The tractor covariant derivative on weight zero tractors

\[
\nabla_\mu T^M := \begin{pmatrix}
\partial_\mu T^+ - T_\mu \\
\nabla_\mu T^m + P^m_\mu T^+ + e^m_\mu T^-\\
\partial_\mu T^- - P^m_\mu T_m
\end{pmatrix} \mapsto U^M_N \nabla_\mu T^N.
\]

Theorem (Sasaki; Bailey, Eastwood, Gover; Nurowski)

\((M, [g])\) conformally Einstein \iff \((M, [g])\) admits a parallel scale tractor.

Proof.

Call \(I_M = (\rho, n_m, \sigma)\) and study

\[
\nabla_\mu I_M = 0 \iff \begin{cases}
\partial_\mu \sigma - n_\mu = 0 \\
\nabla_\mu n_\nu + P_{\mu \nu} \sigma + g_{\mu \nu} \rho = 0 \\
\partial_\mu \rho - P_{\mu \nu} n_\nu = 0
\end{cases}
\]
Physics Dictionary

No physics depends on local choices of unit systems ⇒ express any theory in tractors. Unification a l´a 3 → 4-vectors!

- Einstein–Hilbert action \( S[g, \sigma] = \int \frac{\sqrt{-g}}{\sigma^d} \, l^2 \).
- \( I^M \) parallel ⇒ \( I^2 = constant \); this is the cosmological constant!
- Replace derivatives by Thomas \( D \)-operator; unifies Laplacian and gradient!

\[
D^M := \begin{pmatrix} w(d + 2w - 2) \\ (d + 2w - 2)\nabla \\ -\Delta - wJ \end{pmatrix}, \quad D^M D_M = 0.
\]

- Weights of tractors = masses; Breitenlohner–Freedman bounds for free!

- Wave equations for tractor tensors \( I \cdot D T = 0 \) Laplace–Robin operator

- Example \( T = \varphi \), weight \( w \) scalar, \( \sigma = 1 \),

\[
I \cdot D\varphi = -\left[ \Delta - \frac{2J}{d}w(d + w - 1) \right]\varphi
\]

Mass—Weyl-weight relationship \( m^2 = - \frac{2J}{d} \left[ \left( w + \frac{d-1}{2} \right)^2 - \frac{(d-1)^2}{4} \right] \).

\( \diamond \) Gover, Shaukat, AW
Tractor Maxwell Theory

- Maxwell Tractor, $V^M$, weight $w$ with gauge invariance

$$\delta V^M = D^M \xi = (d + 2w) \begin{pmatrix} (w + 1)\xi \\ \nabla \xi \\ \star \end{pmatrix}$$

- $D_M V^M$ is gauge inert so impose

$$D \cdot V = 0$$

determines $V^-$.

- Get gauge transformations of Stückelberg-massive Proca system

$$\delta V^+ = (d + 2w)(w + 1)\xi$$

$$V^m = (d + 2w)\nabla^m \xi.$$

- Tractor Maxwell Field Strength

$$F^{MN} = D^M V^N - D^N V^M$$

gauge invariant.

- Equations of motion by coupling to scale

$$J^N = I_M F^{MN} = 0$$

Proca Equation
• For $w = -1$, $V^+$ decouples, get standard massless Maxwell.

• For $w = 1 - \frac{d}{2}$, scale tractor decouples:
  
  ▶ In four dimensions $1 - \frac{d}{2} = -1$ so this says Maxwell is Weyl invariant.
  
  ▶ In $d \neq 4$, get Weyl invariant Deser–Nepomechie theory

$$\Delta A_\mu - \frac{4}{d} \nabla^\nu \nabla_\mu A_\nu + \frac{d - 4}{4} \left(2 P_{\mu \nu} A^\nu - \frac{d + 2}{2} A_\mu \right) = 0.$$
Unify Massive, Massless and Partially Massless Theories

Example

Tractor Gravitons \( h_{MN} \) with

\[
\delta h_{MN} = D_M \xi_N + D_N \xi_M, \\
D^M h_{MN} - \frac{1}{2} D_N h^M_M = 0,
\]

Tractor Christoffels \( 2\Gamma_{MNR} = D_M h_{NR} + D_N h_{MR} - D_R h_{MN} \)

\[
G_{MN} = I^R \Gamma_{MNR} = 0, \quad \text{massive gravitons}
\]

- Examine gauge transformations

\[
\begin{cases}
\delta h^{++} = (d + 2w)(w + 1)\xi^+ \\
\delta h^{m+} = (d + 2w)\left[w\xi^m + \nabla^m\xi^+\right] \\
\delta h^{mn} = (d + 2w)\left[\nabla^m\xi^n + \nabla^n\xi^m + \frac{2J}{d}\eta^{mn}\xi^+\right].
\end{cases}
\]

- \( w = 0 \), massless gravitons
- \( w = -1 \), partially massless gravitons

\[
\delta h^{mn} = (d - 2)\left[\nabla^m\nabla^n + \frac{2J}{d}\eta^{mn}\right]\xi^+.
\]
Boundary Problems

Problem

*Given a boundary tractor* $T_\Sigma$, *find a tractor* $T$ *on* $M$ *such that*

$$T|_\Sigma = T_\Sigma \quad \text{and} \quad I \cdot D T = 0.$$  

In a given Weyl frame, this is a Laplace type problem, so *could* just choose coordinates and study the resulting PDE.

Method (formal projector)

- **Boundary data**

- Extend $T_\Sigma$ *arbitrarily* to $T_0$ with $T_0|_\Sigma = T_\Sigma$

- Iteratively find $T^{(1)}$, $T^{(2)}$, ... approaching solution $T$, with $T^{(l)}|_\Sigma = T_\Sigma$

- Check solution $T$ is *independent* of original choice of extension $T_0$.  

A. Waldron (Davis)
Boundary Calculus

Observe that because $T|_{\Sigma} = T_{\Sigma} = T_0|_{\Sigma}$

$$T = T_0 + \sigma S,$$

for some $S$.

This suggests to search for an expansion in the scale!

$$T^{(l)} = T_0 + \sigma T_1 + \sigma^2 T_2 + \cdots + \sigma^l T_l.$$

Need algebra of $l \cdot D$ and $\sigma$, remarkably

$$[l \cdot D, \sigma] = (d + 2w)l^2$$

Or calling $x := \sigma$, $h := d + 2w$, $y = -\frac{1}{l^2} l \cdot D$ we have the $sl(2)$ solution generating algebra

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y$$
The Solution

Example

Given weighted tractor $hT = h_0 T$ and $T = T_0 + xT_1 + x^2 T_2 + \cdots$ then

$$yT = yT_0 - (h_0 - 2) T_1 + x(yT_1 - 2(h_0 - 3) T_2) + \cdots,$$

so

$$T = T_0 + \frac{1}{h_0 - 2} x yT_0 + \frac{1}{2(h_0 - 2)(h_0 - 3)} x^2 y^2 T_0 + \cdots.$$

All order solution given by solution generating operator

$$T = :K(z): T_0$$

with

$$K(z) = z^{\frac{h_0 - 1}{2}} \Gamma(2 - h_0) J_{1-h_0}(2\sqrt{z}) = 1+ \frac{1}{h_0 - 2} z + \frac{1}{2(h_0 - 2)(h_0 - 3)} z^2 + \cdots$$

and $z = xy$ with normal ordering

$$: z^k : = x^k y^k.$$
Tangential Operators

We call a bulk operator $\mathcal{O}$ *tangential* if

$$\mathcal{O}\sigma = \sigma\mathcal{O}' , \quad \text{for some } \mathcal{O}'$$

**Example**

- The tangential derivative $\nabla^T := \nabla - n\nabla_n$

- The solution generating operator $: K :$ obeys $: K : x = 0$

  for same reason that $y : K : \equiv 0$.

- The *holographic GJMS operator* $y^k$, $k \in 2\mathbb{N}$

  acting on tractors with weight $h_0 = k + 1$, because

  $$[x, y^k] = y^{k-1}k(h - k + 1).$$

When bulk operators are tangential they define boundary operators since $\mathcal{O}T|_\Sigma$ is independent of how $T_\Sigma$ is extended to $T$. 
Obstructions and Anomalies

When \( h_0 = 2, 3, \ldots \), so \( w + \frac{d}{2} = 1, \frac{3}{2}, 2, \ldots \), the recurrence

\[
T_k = \frac{1}{k(h_0 - k - 1)} y T_{k-1}
\]

fails for \( T_{h_0-1} \).

N.B., usually \( T_k \sim y^k T_0 \), so the operator \( y^{h_0-1} \) is the obstruction.

- For conformally Einstein bulk and \( h_0 = 2, 4, 6, \ldots \), \( y^k \) vanishes.
- For \( h_0 = 3, 5, 7, \ldots \) the tangential operator \( y^k \) is a holographic formula for the GJMS operator

\[
P_{2k} = \Delta^k + \text{curvatures}.
\]

Conformally invariant boundary operators corresponding to conformal anomalies.

- Acting on log densities\(^1\) \( y^{d-1} \) yields Branson’s \( Q \)-curvature

\[
Q = \frac{1}{((d - 2)!!)^2} y^{d-1} U|\Sigma.
\]

Holographic anomalies of Henningson–Skenderis.

\(^1\)Under conformal transformations \( U \mapsto U + \log \Omega \).
Log solutions

Using \([y, x^k] = -x^{k-1}k(h + k - 1)\) we learn

\[
y : K(z) : = : \left( zK''(z) + K'(z)(2 - h) + K(z) \right) : y ,
\]

Operator problem now an ODE—Bessel type-equation solvable by Frobenius method:

- Second solution = \(z^{h_0 - 1}(\text{first solution } h_0 \rightarrow 2 - h_0)\)
- \(h_0 \in \mathbb{N}\), Log solution = (degree \(h_0 - 2\) polynomial) + \(z^{h_0 - 1} \log z \times (\text{second solution}) + \text{“finite terms”}\)
- Log solution requires second scale \(\tau\), at definite weight only \(\log(\sigma/\tau)\) can appear. \(\tau|_{\Sigma} \neq 0\)
- \(\log z\) is completely formal because \(z = xy\), but \(\log \tau\) can play rôle of “\(\log y\)”.
- Algebra of \(\log x\),

\[
[y, \log x] = -\frac{1}{x}(h - 1)
\]

- Must also require solution generating operator to be tangential.
The Solution

Remarkably, can solve to all orders at log weights

\[ T = \mathcal{O} T_0, \quad \text{solves } I \cdot D T = 0 \]

\[ \mathcal{O} = : F_{h_0 - 2}(z) : - : z^{h_0} B(z) : \frac{1}{(h_0 - 1)! (h_0 - 2)!} \]

\[ x^{h_0 - 1} \log x : K_{h_0}(z) : y^{h_0 - 1} - x^{h_0 - 1} : K_{h_0}(z) : \left[ \log \tau \, y^{h_0 - 1} \right]_W \]

- \( F_{h_0 - 2} = 1+\cdots \) is the standard solution up to orders before obstruction—“infinities”.
- log terms multiply second solution \( K_{h_0} = 1+\cdots \)
- careful Weyl ordering of operators \( y \) and \( \log \tau \) ensures tangentiality \( \mathcal{O} x = 0 \).
- \( B = 1+\cdots \) are non-log finite terms. Explicit formulae for all terms.

Solution of wave equation boundary problem for arbitrary tensors in any curved bulk.
Ambient Tractors

Flat model for conformal manifold

- Ambient space $\tilde{M} = \mathbb{R}^{d+1,1}$
- Lightcone $Q = \{X^M X_M = 0\}$
- Conformal manifold $M = \{\text{lightlike rays}\}$
Momentum Cone

Tractors are equivalence classes of weighted ambient tensors $T$ (Gover, Peterson, Čap)

$$T \sim T + X^2 S, \quad X^M \nabla_M T = w T.$$  

Tractor operators respect equivalence classes

$$\mathcal{O} X^2 = X^2 \mathcal{O}'$$

Fundamental operators $\leftrightarrow$ momentum space representation of the ambient conformal group (Gover, AW)

<table>
<thead>
<tr>
<th>Canonical Tractor</th>
<th>$X^M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>$w = \nabla_X$</td>
</tr>
<tr>
<td>Double $D$-operator</td>
<td>$D_{MN} = X_N \nabla_M - X_M \nabla_N$</td>
</tr>
<tr>
<td>Thomas $D$-operator</td>
<td>$D_M = \nabla_M (d + 2 \nabla_X - 2) - X_M \Delta$</td>
</tr>
</tbody>
</table>
Curved Cone

Metric on curved ambient space \((\tilde{M}, g_{MN})\):

\[
g_{MN} = \nabla_M X_N
\]

where \(X\) is now \textit{hypersurface orthogonal homothetic} vector field.

Consequences:

\[
\mathcal{L}_X g_{MN} = 2g_{MN}, \quad \nabla [M X_N] = 0, \quad X_M = \nabla_M \frac{1}{2} X^2, \quad g_{MN} = \frac{1}{2} \nabla_M \partial_N X^2.
\]

So \(X^2\) is \textit{homothetic potential} and defines a \textit{curved cone}.

Define tractors as before \(\Rightarrow\) arbitrary curved space.

Remarkably have an \(sl(2) \cong sp(2)\) algebra from operators (GJMS)

\[
\mathcal{Q} = \begin{pmatrix} X^2 & \nabla X + \frac{d+2}{2} \Delta \\ \nabla X + \frac{d+2}{2} \Delta & \Delta \end{pmatrix}, \quad [Q_{ij}, Q_{kl}] = \epsilon_{kj} Q_{ik} + \text{(3 more)}
\]
Two Times Physics

Itzhak Bars:

$$H \rightarrow Q_{ij}, \quad \mathbb{R}^{d-1,1} \rightarrow \mathbb{R}^{d,2}$$

because *Howe dual pair*

$$\mathfrak{sp}(2) \otimes \mathfrak{so}(d,2) \subset \mathfrak{sp}(2(d + 2))$$

Particle action

$$S = \int dt \left[ P_M \dot{X}^M - \lambda^{ij} Q_{ij} \right], \quad Q = \begin{pmatrix} X^2 & X \cdot P \\ X \cdot P & P^2 \end{pmatrix} \iff \begin{cases} \text{relativistic particle} \\
\text{AdS particle} \\
\text{H-atom} \\
\text{Harmonic Oscillator} \\
\vdots \end{cases}$$

Bars proposed

Gravity $\leftrightarrow \left\{ \begin{array}{l}
\text{triplets of Hamiltonians in } 2(d + 2) \text{ dimensional phase space} \\
\text{obeying } \mathfrak{sp}(2) \text{ algebra}
\end{array} \right\}$

Confirm this proposal using tractors! (Bonezzi, Latini, AW)
Gravity

\[\begin{align*}
[Q_{ij}, Q_{kl}] &= \epsilon_{kj} Q_{il} + \epsilon_{ki} Q_{jl} + \epsilon_{lj} Q_{ik} + \epsilon_{li} Q_{jk} \\
Q\psi &= 0
\end{align*}\]

- **Symplectic Gauge Invariance**
  \[Q \mapsto Q + [Q, \epsilon], \quad \psi \mapsto \psi + \epsilon \psi\]

- **Expand** \(Q, \epsilon\) in powers of operator \(\nabla\) \(\rightarrow\) infinitely many fields

- **Solve** \(s\ell(2)\) conditions

\[Q = \begin{pmatrix}
X^M G_{MN} X^N & X^M (\nabla_M + A_M) + \frac{d+2}{2} \\
X^M (\nabla_M + A_M) + \frac{d+2}{2} & (\nabla_M + A^M) G_{MN} (\nabla^N + A^N)
\end{pmatrix}, \quad \epsilon = \alpha + \xi^M (\nabla_M + A_M), \quad G_{MN} = \nabla_M X_N, \quad X^M F_{MN}(A) = 0.\]
Many Actions

- Lagrange multipliers for Hamiltonian constraints

\[ S(G_{MN}, A_M, \Psi, \Omega, \Theta, \Lambda) = \int \sqrt{G} \left( \Omega \tilde{\nabla}^2 + \Theta [X.\tilde{\nabla} + \frac{d+2}{2}] + \Lambda X^2 \right) \Psi \]

- \( \Theta \) fixes weight \( \nabla_X \Psi = (w - \frac{d}{2} - 1)\Psi \)
- \( \Lambda \) says \( \Psi = \delta(X^2)\phi \) so \( \phi \sim \phi + X^2\chi \)
- \( \Rightarrow S = \int \sqrt{G} \delta(X^2) T(G, A, \Omega, \phi) \)
- \( T = \phi(\nabla + A)^2 \Omega \) must be a tractor: in Maxwell gauge \( X.A = w \)

\[ T = \phi \left( \frac{1}{w} A^M D_M - \frac{1}{d-2} (D^M A_M) + A^2 \right) \Omega \]

- \( T \) tractor \( \Rightarrow d \)-dimensional action \( S = \int \sqrt{-g} T \)
- Residual \( SO(1, 1) \) gauge invariance

\[ \delta \Omega = \alpha \Omega, \quad \delta \phi = -\alpha \phi, \quad \delta A^M = \frac{1}{d-2} D^m \alpha \]
Singlet $\Omega \phi =: \varphi^2$ is gauge invariant.

Integrate out $A_M$ leaves only $\varphi$ and metric

\[ S = \int \sqrt{-g} \varphi \left[ \Delta - \frac{d-2}{2} J \right] \varphi \]

CONFORMALLY IMPROVED SCALAR

In terms of scale

\[ \varphi = \sigma^{1-\frac{d}{2}} \]

Tractor Einstein–Hilbert action

\[ S = \int \frac{\sqrt{-g}}{\sigma^d} l^2 \]

Weyl invariant

Choose $\sigma = 1,$

\[ S = \int \sqrt{-g} R. \]
Conclusions and Outlook

- Transversal tensors ✓
- Ubiquity of $I \cdot D$, harmonic Weyl tensor $\rightarrow I.\nabla W_{MNRS} = 0$ for Weyl tractor.
- Global solution?
- Correlator calculus.
- Two times and dualities.