

# The renormalization group and Weyl invariance

Roberto Percacci

SISSA Trieste

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## FRGE I: DEFINITIONS

$$e^{-W_k[J]} = \int (D\phi) \exp(-(S + \Delta S_k + \int J\phi))$$

$$\Delta S_k(\phi) = \frac{1}{2} \int d^4q \phi(-q) R_k(q^2) \phi(q)$$

with  $R_k(q^2) \rightarrow 0$  for  $k \rightarrow \infty$ ,  $R_k(q^2) \rightarrow k^2$  for  $k \rightarrow 0$

$$\Gamma_k[\phi] = W_k[J] - \int J\phi - \Delta S_k$$

note  $\lim_{k \rightarrow 0} \Gamma_k = \Gamma$

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + R_k \right)^{-1} \partial_t R_k$$

C. Wetterich, Phys. Lett. B **301** (1993) 90.

Trace is over momenta and all quantum numbers

## FRGE III: ONE LOOP

$$\Gamma^{(1)} = S + \frac{1}{2} \ln \det \mathcal{O} ; \quad \mathcal{O} = \frac{\delta^2 S}{\delta\phi\delta\phi}$$

$$\Gamma_k^{(1)} = S + \Delta S_k + \frac{1}{2} \ln \det \mathcal{O}_k - \Delta S_k ; \quad \mathcal{O}_k = \frac{\delta^2 (S + \Delta S_k)}{\delta\phi\delta\phi} = \mathcal{O} + R_k$$

$$\partial_t \Gamma_k^{(1)} = \frac{1}{2} (\det \mathcal{O}_k)^{-1} \partial_t \det \mathcal{O}_k = \frac{1}{2} \text{Tr} \mathcal{O}_k^{-1} \partial_t \mathcal{O}_k$$

One-loop RG equation

$$\partial_t \Gamma_k^{(1)} = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 S}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k$$

## FRGE IV: BETA FUNCTIONS

$$\Gamma_k(\phi) = \sum_i g_i(k) \mathcal{O}_i(\phi)$$

$$\partial_t \Gamma_k = \sum_i \partial_t g_i \mathcal{O}_i = \sum_i \beta_{g_i} \mathcal{O}_i$$

compare with

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + R_k \right)^{-1} \partial_t R_k$$

use to calculate  $\Gamma$

or to investigate the UV properties of the theory

## Dimensional analysis

Choose dimensionless coordinates.

Invariance under global Weyl transformations

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \psi_a \rightarrow \Omega^{d_a} \psi_a, g_i \rightarrow \Omega^{d_i} g_i$$

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{d_a} \psi_a, \Omega^{d_i} g_i)$$

fixes  $d_a, d_i$ . *ALWAYS* true.

Global scale transformations

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{d_a} \psi_a, g_i)$$

not always true, must be  $d_i = 0$ .

## Weyl's idea

Interpret scale transformations as changes of the unit of length.

Allow choice of unit to depend on position.

Allow parallel transport to affect norm of vectors.

“It is evident that two rods side by side, stationary with respect to each other, can be intercompared....this cannot be done for....rods with either a space- or time-like separation”. (Dicke 1962)

Physics must be formulated in a way that is invariant under local changes of units, i.e. under *local* rescalings of the metric.

## Weyl gauging

Abelian gauge field  $\kappa_\mu \mapsto \kappa_\mu + \Omega^{-1} \partial_\mu \Omega$   
scalar field  $\phi$  transforming as  $\phi \rightarrow \Omega^d \phi$

$$D_\mu \phi = \partial_\mu \phi - d \kappa_\mu \phi$$

More generally

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu - \delta_\mu^\lambda \kappa_\nu - \delta_\nu^\lambda \kappa_\mu + g_{\mu\nu} \kappa^\lambda$$

is invariant under local Weyl transformations, hence for a tensor of dimension  $d$

$$D_\mu t = \hat{\nabla}_\mu t - d \kappa_\mu t$$

is diffeomorphism and Weyl covariant.



## Weyl curvature

$$[D_\mu, D_\nu]v^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma v^\sigma$$

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + (w - 1)F_{\mu\nu}g_{\rho\sigma} \\ &\quad + g_{\mu\rho}(\nabla_\nu\kappa_\sigma + \kappa_\nu\kappa_\sigma) - g_{\mu\sigma}(\nabla_\nu\kappa_\rho + \kappa_\nu\kappa_\rho) \\ &\quad - g_{\nu\rho}(\nabla_\mu\kappa_\sigma + \kappa_\mu\kappa_\sigma) + g_{\nu\sigma}(\nabla_\mu\kappa_\rho + \kappa_\mu\kappa_\rho) - (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\kappa^2\end{aligned}$$

$$F_{\mu\nu} = \partial_\mu\kappa_\nu - \partial_\nu\kappa_\mu$$

## Weyl's failed unification idea

Weyl regarded  $\kappa_\mu$  as e.m. field.

Einstein's critique: spectral lines would be blurred.

Weyl's theory still viable if  $\kappa_\mu$  interpreted as a piece of the spacetime connection and its curvature is sufficiently weak.

Assume curvature of  $\kappa_\mu$  is zero. Then

$$\kappa_\mu = -\chi^{-1} \partial_\mu \chi$$

where the field  $\chi$  transforming as  $\chi \rightarrow \Omega^{-1} \chi$  is....

## ...the dilaton

Starting from any action  $S(g_{\mu\nu}, \psi_a, g_i)$ , replace  $g_i \rightarrow \chi^{d_i} \hat{g}_i$ ,  $\nabla \rightarrow D$  and all curvatures  $R \rightarrow \mathcal{R}$ .

Resulting action  $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$  is Weyl invariant and agrees with original action in the gauge where  $\chi$  is constant

$$\hat{S}(g_{\mu\nu}, \chi_0, \psi_a, \hat{g}_i) = S(g_{\mu\nu}, \psi_a, g_i)$$

Dilaton acts as *classical* compensator field (Stückelberg)

## Motivation: scalar-tensor theory

$$\int d^4x \sqrt{g} \left[ \frac{1}{2} (\nabla\phi)^2 + V(\phi^2) + F(\phi^2)R \right]$$

$$V(\phi^2) = \lambda_0 + \lambda_2\phi^2 + \lambda_4\phi^4 + \dots$$

$$F(\phi^2) = \xi_0 + \xi_2\phi^2 + \dots$$

$$\int d^4x \sqrt{g} \left[ \lambda_4\phi^4 + \frac{1}{2} (\nabla\phi)^2 + \xi_2\phi^2 R \right]$$

Weyl-invariant for  $\xi_2 = \frac{1}{12}$ .

Not preserved by RG flow D. Perini and R.P. Phys. Rev. **D68** 044018 (2003) hep-th/0304222; G. Narain and R.P. Class. Quant. Grav. **27** 075001 (2010) arXiv:0911.0386 [hep-th])

## The conformal anomaly

Cutoff breaks scale invariance, but anomaly can be avoided in theories with dilaton

Englert, C. Truffin and R. Gastmans, “Conformal invariance in quantum gravity”, Nucl. Phys. B117, 407 (1976)

R.Floresanini and R. P., “Average effective potential for the conformal factor”, Nucl. Phys. B436, 141 (1995)

M. Shaposhnikov and I. Tkachev, “Quantum scale invariance on the lattice”, Phys. Lett. B675, 403 (2009)

## Example: matter in background metric and dilaton

$$S(\phi, g_{\mu\nu}) = \frac{1}{2} \int d^4x \sqrt{g} \phi \Delta^{(1/6)} \phi, \quad \Delta^{(1/6)} = -\square + \frac{R}{6}$$

$$S(\psi, g_{\mu\nu}) = \int d^4x \sqrt{g} \bar{\psi} D\psi$$

$$S(A, g_{\mu\nu}) = \frac{1}{4} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu}$$

## Trace anomaly

$$\langle T^\mu{}_\mu \rangle = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta\Gamma}{\delta g^{\mu\nu}} = b C^2 + b' E$$

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

$$C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

$$b = \frac{1}{120(4\pi)^2} (n_S + 6n_D + 12n_M)$$

$$b' = -\frac{1}{360(4\pi)^2} (n_S + 11n_D + 62n_M)$$

## The scalar and fermion measure

$$(d\phi) = \prod_x \frac{d\phi(x)}{\mu}$$

$$\Gamma(g_{\mu\nu}) = \int (d\phi) e^{-\int d^4x \sqrt{g} \phi \Delta^{(1/6)} \phi} = \frac{1}{2} \ln \det \left( \frac{\Delta^{(1/6)}}{\mu^2} \right)$$

$$(d\psi) = \prod_x \frac{d\psi(x)}{\mu^{3/2}}$$



## The Maxwell theory measure

$$(dA_\mu) = \prod_x dA_\mu(x)$$

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g} (\nabla^\mu A_\mu)^2$$

$$S + S_{GF} = \frac{1}{2} \int d^4x \sqrt{|g|} A_\mu \left( -\nabla^2 g^{\mu\nu} + R^{\mu\nu} \right) A_\nu$$

$$S_{gh} = \int d^4x \sqrt{|g|} \bar{C} (-\nabla^2) C$$

$$\begin{aligned} \Gamma(g) &= \log \int (dA d\bar{C} dC) e^{-S_{em}(A,g) - S_{GF}(A,g) - S_{gh}(\bar{C},C,g)} \\ &= \frac{1}{2} \text{Tr} \left( \frac{\log \Delta^{(1)}}{\mu^2} \right) - \text{Tr} \log \left( \frac{\Delta^{(0)}}{\mu^2} \right) \end{aligned}$$

## The Weyl-invariant scalar measure

$$(d\phi) = \prod_x d\left(\frac{\phi(x)}{\chi(x)}\right)$$

$$\Gamma(g_{\mu\nu}) = \int (d\phi) e^{-\int d^4x \sqrt{g} \phi \Delta^{(1/6)} \phi} = \frac{1}{2} \ln \det \left( \frac{1}{\chi^2} \Delta^{(1/6)} \right)$$

$$\frac{1}{\Omega^{-2} \chi^2} \Delta_{\Omega^2 g}^{(1/6)} (\Omega^{-1} \phi) = \Omega^{-1} \left( \frac{1}{\chi^2} \Delta^{(1/6)} \phi \right)$$

eigenvalues are Weyl invariant  $\rightarrow \det \left( \frac{1}{\chi^2} \Delta^{(1/6)} \right)$  is Weyl invariant

## Weyl-covariant gauge fixing

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g} (D^\mu A_\mu)^2$$

$$S + S_{GF} = \frac{1}{2} \int d^4x \sqrt{g} A_\mu \left( -D^2 g^{\mu\nu} + R^{\mu\nu} \right) A_\nu$$

eigenvalues of  $-D^2 \delta_\nu^\mu + R_\nu^\mu$  are Weyl invariant

$\rightarrow \det(-D^2 \delta_\nu^\mu + R_\nu^\mu)$  is Weyl invariant

## Weyl invariant quantization

dilaton acts as compensator (Stückelberg) field in the *quantum* effective action

$\mu$  has been promoted to a field

## Another point of view

$$\Gamma^I(g^\chi) - \Gamma^I(g) = \Gamma_{WZ}(g, \chi)$$

Wess-Zumino consistency condition:

$$\Gamma_{WZ}(g^\Omega, \chi^\Omega) - \Gamma_{WZ}(g, \chi) = -\Gamma_{WZ}(g, \Omega)$$

where  $g^\Omega = \Omega^2 g$ ,  $\chi^\Omega = \Omega^{-1} \chi$

If we identify  $\Gamma^I(g)$  with  $\Gamma^{\text{II}}(g, \chi = \mu)$ ,

$$\Gamma^{\text{II}}(g, \chi) = \Gamma^I(g) + \Gamma_{WZ}(g, \chi)$$

## Dynamical metric and dilaton

$$S = \int d^4x \sqrt{g} \left[ \lambda Z^2 \chi^4 - \frac{1}{2} Z \left( \xi \chi^2 R + g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right) \right]$$

for  $\xi = 1/6$

$$S = \int d^4x \sqrt{g} \left[ \lambda Z^2 \chi^4 - \frac{1}{12} Z \chi^2 \mathcal{R} \right]$$

can choose Weyl gauge (i.e. units) where

$$Z \chi^2 = \frac{12}{16\pi G}; \quad \lambda = \frac{2\pi}{9} G \Lambda$$

$S$  reduces to Hilbert action.

For  $\xi \neq 1/6$  scalar field is physical.

## Expansion of action

Let  $\delta g_{\mu\nu} = h_{\mu\nu}$ ,  $\delta\chi = \eta$ .

$$\begin{aligned} \mathcal{S}^{(2)} &= \frac{1}{2} Z \mathcal{H}((h, \eta), (h, \eta)) \\ &= \frac{1}{2} Z \int d^4x \sqrt{g} (h_{\mu\nu} \quad \eta) \begin{pmatrix} \mathcal{H}_{hh}^{\mu\nu\rho\sigma} & \mathcal{H}_{h\eta}^{\mu\nu} \\ \mathcal{H}_{\eta h}^{\rho\sigma} & \mathcal{H}_{\eta\eta} \end{pmatrix} \begin{pmatrix} h_{\rho\sigma} \\ \eta \end{pmatrix} \end{aligned}$$

For  $\xi = 1/6$

$$\begin{aligned}\mathcal{H}_{hh}^{\mu\nu\rho\sigma} = & \frac{1}{12}\chi^2 \left[ -\frac{1}{2}\mathbf{1}^{\mu\nu\rho\sigma} D^2 + g^{(\nu|\sigma} D^{|\mu)} D^\rho - \frac{1}{2}g^{\mu\nu} D^\rho D^\sigma - \frac{1}{2}g^{\rho\sigma} D^{(\mu} D^{\nu)} \right. \\ & + \frac{1}{2}g^{\mu\nu} g^{\rho\sigma} D^2 - \mathcal{R}^{\mu\rho\nu\sigma} - g^{(\mu|\rho} \mathcal{R}^{|\nu)\sigma} + \frac{1}{2}(g^{\mu\nu} \mathcal{R}^{\rho\sigma} + \mathcal{R}^{\mu\nu} g^{\rho\sigma}) \\ & \left. + (\mathcal{R} - 12\lambda Z \chi^2) K^{\mu\nu\rho\sigma} \right]\end{aligned}$$

$$\mathcal{H}_{h\eta}^{\mu\nu} = \mathcal{H}_{\eta h}^{\mu\nu} = \frac{1}{6}\chi \left( g^{\mu\nu} D^2 - D^\mu D^\nu + \mathcal{R}^{\mu\nu} - \frac{1}{2}\mathcal{R}g^{\mu\nu} \right) + 2\lambda Z \chi^3 g^{\mu\nu}$$

$$\mathcal{H}_{\eta\eta} = D^2 - \frac{1}{6}\mathcal{R} + 12\lambda Z \chi^2 .$$

background Weyl transformations

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu} ; \quad \chi \mapsto \Omega^{-1} \chi ; \quad h_{\mu\nu} \mapsto \Omega^2 h_{\mu\nu} ; \quad \eta \mapsto \Omega^{-1} \eta$$



## Wave operators

$$\mathcal{G}((h_1, \eta_1), (h_2, \eta_2)) = \int d^4x \sqrt{g} \left[ \chi^4 h_{1\mu\nu} g^{\mu\rho} g^{\nu\sigma} h_{2\rho\sigma} + \chi^2 \eta_1 \eta_2 \right]$$

$$\mathcal{S}^{(2)} = \frac{1}{2} Z \mathcal{H}(\theta, \theta) = \frac{1}{2} Z \mathcal{G}(\theta, \mathcal{O}\theta)$$

$$(\mathcal{O}_{hh})_{\mu\nu}{}^{\rho\sigma} = \chi^{-4} g_{\mu\alpha} g_{\nu\beta} \mathcal{H}_{hh}^{\alpha\beta\rho\sigma},$$

$$(\mathcal{O}_{h\eta})_{\mu\nu} = \chi^{-4} g_{\mu\alpha} g_{\nu\beta} \mathcal{H}_{h\eta}^{\alpha\beta},$$

$$\mathcal{O}_{\eta h}^{\rho\sigma} = \chi^{-2} \mathcal{H}_{\eta h}^{\rho\sigma},$$

$$\mathcal{O}_{\eta\eta} = \chi^{-2} \mathcal{H}_{\eta\eta}.$$

## Weyl covariance of wave operators

$$\begin{aligned}(\mathcal{O}_{hh}(\Omega^2 g_{\mu\nu}, \Omega^{-1} \chi))_{\mu\nu}{}^{\rho\sigma}(\Omega^2 h_{\rho\sigma}) &= \Omega^2 (\mathcal{O}_{hh}(g_{\mu\nu}, \chi))_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} , \\(\mathcal{O}_{h\eta}(\Omega^2 g_{\mu\nu}, \Omega^{-1} \chi))_{\mu\nu}(\Omega^{-1} \eta) &= \Omega^2 (\mathcal{O}_{h\eta}(g_{\mu\nu}, \chi))_{\mu\nu} \eta , \\(\mathcal{O}_{\eta h}(\Omega^2 g_{\mu\nu}, \Omega^{-1} \chi))^{\rho\sigma}(\Omega^2 h_{\rho\sigma}) &= \Omega^{-1} (\mathcal{O}_{\eta h}(g_{\mu\nu}, \chi))^{\rho\sigma} h_{\rho\sigma} , \\ \mathcal{O}_{\eta\eta}(\Omega^2 g_{\mu\nu}, \Omega^{-1} \chi)(\Omega^{-1} \eta) &= \Omega^{-1} \mathcal{O}_{\eta\eta}(g_{\mu\nu}, \chi) \eta .\end{aligned}$$

## Gauge fixing

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g} \frac{1}{2} Z_\xi \chi^2 F_\mu \bar{g}^{\mu\nu} F_\nu ,$$

where

$$F_\nu = D_\mu h^\mu{}_\nu - \frac{\beta + 1}{4} D_\nu h$$

$$S_{gh} = \mathcal{G}_{gh}(\bar{C}, \mathcal{O}_{gh} C) .$$

$$\mathcal{G}_{gh}(A, B) = \int d^4x \sqrt{g} \chi^2 A_\mu g^{\mu\nu} B_\nu$$

$$(\mathcal{O}_{gh})_\mu^\nu = -\frac{1}{\chi^2} \left( \delta_\mu^\nu D^2 + \frac{1-\beta}{2} D_\mu D^\nu + \mathcal{R}_\mu{}^\nu \right)$$

Weyl-gauge-fixing  $\eta = 0$

## Weyl invariance of one loop effective action

$$\Gamma^{(1)}(g_{\mu\nu}, \chi) = S(g_{\mu\nu}, \chi) + \frac{1}{2} \text{Tr} \log \mathcal{O}_{hh} - \text{Tr} \log \mathcal{O}_{gh}$$

If  $\Delta_{(g_{\mu\nu}, \chi)} h_{\mu\nu} = \lambda h_{\mu\nu}$  then

$$\Delta_{(\Omega^2 g_{\mu\nu}, \Omega^{-1} \chi)} (\Omega^2 h_{\mu\nu}) = \Omega^2 \Delta_{(g_{\mu\nu}, \chi)} h_{\mu\nu} = \lambda \Omega^2 h_{\mu\nu}$$

The spectrum of  $\Delta$  is Weyl invariant, so  $\Gamma^{(1)}$  is Weyl invariant.

$$\Delta S_k = \frac{1}{2} Z \mathcal{G} \left( h, \frac{1}{12} \frac{1}{\chi^2} R_k(\chi^2 \mathcal{O}_{hh}) h \right) + \mathcal{G}_{gh} \left( \bar{C}, \frac{1}{\chi^2} R_k(\chi^2 \mathcal{O}_{gh}) C \right)$$

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} F(\mathcal{O}_{hh}) - \text{Tr} F(\mathcal{O}_{gh})$$

r.h.s. of FRGE is Weyl invariant if initial point is Weyl invariant  
 also  $\Gamma$  is Weyl invariant

## Beta functions

Example:

$$\Gamma_k = \int d^4x \sqrt{g} \left[ \lambda Z^2 \chi^4 - \frac{1}{12} Z \chi^2 \mathcal{R} \right]$$

$$u \frac{dZ}{du} = \frac{23}{4\pi^2} u^2$$

$$u \frac{d\lambda}{du} = \frac{u^2}{16\pi^2 Z^2} (u^2 - 184Z\lambda)$$

where  $u = k/\chi$  is assumed constant

## Solutions

$$Z(u) = Z_0 + \frac{23}{8\pi^2} u^2 \rightarrow \frac{23}{8\pi^2} u^2$$

$$\lambda(u) = \frac{\pi^2(u^4 + 64\pi^2 Z_0^2 \lambda_0)}{(8\pi^2 Z_0 + 23u^2)^2} \rightarrow \frac{\pi^2}{529}$$

note:  $Z$  is redundant coupling and does not reach a FP,  
 $\lambda$  is essential coupling and reaches a FP.

$$\tilde{m}_P^2 = Z \frac{\chi^2}{k^2} = \frac{Z}{u^2} \rightarrow \frac{23}{8\pi^2}$$

## Conclusions

- in presence of a dilaton, it is possible to quantize theory preserving Weyl invariance.
- RG flow also preserves Weyl invariance.
- $\xi = 1/6$  is a fixed point of RG flow.
- cutoff and renormalization scale can depend on position. Theory with nonconstant cutoff equivalent to theory with constant cutoff but conformally related metric.
- generalize to nonflat Weyl connection