Scale vs Conformal invariance from holography

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Scale invariance
= Conformal invariance?
Scale = Conformal?

• QFTs and RG-groups are classified by scale invariant IR fixed point (Wilson’s philosophy)

• Conformal invariance gave a (complete?) classification of 2D critical phenomena

• But scale invariance does not imply conformal invariance???
In equations...

- Scale invariance
  \[ x^\mu \rightarrow \lambda x^\mu \]
  → Trace of energy-momentum (EM) tensor is a divergence of a so-called **Virial current**
  \[ T^{\mu}_{\mu} = \partial^\mu J_\mu \quad D_\mu = x_\nu T^\nu_\mu - J_\mu \]

- Conformal invariance
  \[ x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2a^\mu x_\mu + a^2 x^2} \]

- EM tensor can be improved to be **traceless**
  \[ J_\mu = \partial^\nu L_{\mu\nu} \quad T^\mu_\mu \rightarrow \tilde{T}^\mu_\mu = 0 \]
In today’s talk

- I’ll argue for the equivalence between scale and conformal from holographic viewpoint
- Importance of full diffeomorphism
- How to construct scale but non-conformal holographic models
- Surprising trace anomaly from holography
Part 1. From field theory
Maxwell theory in \( d > 4 \)

- Scale invariance does \textbf{NOT} imply conformal invariance in \( d > 4 \) dimension.

- \((d>4)\) \textbf{free Maxwell theory} is an example (Jackiw and Pi, El-Showk, Nakayama and Rychkov)

- It is an isolated example because one cannot introduce non-trivial interaction
Maxwell theory in $d > 4$

- EM tensor and Virial current
  \[ T_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{g_{\mu\nu}}{4}(F_{\rho\sigma}F^{\rho\sigma}) \]
  \[ T_\mu^\mu = \frac{4-d}{4} F_{\mu\nu}F^{\mu\nu} = \frac{4-d}{4} \partial_\mu(F^{\mu\rho}A_\rho) \]

- Virial current $J^\mu = F^{\mu\rho}A_\rho$ is not a derivative so one cannot improve EM tensor to be traceless

- Dilatation current is not gauge invariant, but the charge is gauge invariant

- Does not satisfy Ward identity for conformal inv

- You may take a very specific gauge so that the gauge fixed action is conformal, but conformal symmetry does not commute with BRST
Zamolodchikov-Polchinski theorem (1988):
A scale invariant field theory is conformal invariant in (1+1) d when

1. It is unitary
2. It is Poincare invariant (causal)
3. It has a discrete spectrum
4. Scale invariant current exists
(1+1) d proof

\[ F(x^2) = z^4 \langle T(x)T(0) \rangle \quad T \equiv T_{xx}, \quad \Theta \equiv T_{\mu}^\mu \]

\[ G(x^2) = z^3 \bar{z} \langle \Theta(x)T(0) \rangle \quad \bar{\partial}T + 4\partial\Theta = 0 \]

\[ H(x^2) = z^2 \bar{z}^2 \langle \Theta(x)\Theta(0) \rangle \]

According to Zamolodchikov, we define

\[ C = 2(F - \frac{1}{2}G - \frac{3}{16}H) \]

\[ \dot{C} \equiv x^2 \frac{d}{dx^2}C = -\frac{3}{4}H \leq 0 \quad \text{← C-theorem!} \]

At RG fixed point, \( \dot{C} = 0 \), which means \( H = 0 \)

\( \Rightarrow \quad \langle \Theta(x)\Theta(0) \rangle = 0 \quad \iff \quad \Theta = 0 \)
Reeh-Schlieder theorem
\[ \langle \Theta(x)\Theta(0) \rangle = 0 \iff \Theta = 0 \]

In CFT, this is true by state-operator correspondence. In generic QFT, it is due to the unitarity and Reeh-Schlieder theorem.

Unitarity tells you
\[ \Theta(x)|0\rangle = 0 \]

We then use Reeh-Schlieder theorem
\[ O(x)|0\rangle = 0 \iff O(x) = 0 \]

Proof is not that elementary: we need to use microcausality, so it is intrinsic to relativistic field theory (in Schrodinger theory, it is not true).
Hofman-Strominger theorem(?) (2011):
A chiral scale invariant field theory is chiral conformal invariant in (1+1) d when

1. It is unitary
2. It has a discrete spectrum
3. Symmetry currents exist

But presumably (??) without Lorentz inv

\[ H \left( L_1 \right): t \rightarrow t + \epsilon_t \quad D \left( L_0 \right): t \rightarrow \lambda t, \quad x \rightarrow x \]
\[ P \left( \bar{L}_1 \right): x \rightarrow x + \epsilon_x \]

Then chiral conformal (and actually chiral Virasoro) appears
\[ K \left( L_{-1} \right): t \rightarrow t + \epsilon t^2 \]
Proof ??

\[ \partial_x T_{tx} + \partial_t T_{xx} = 0 \]
\[ \partial_x T_{tt} + \partial_t T_{xt} = 0 \]

D + locality \rightarrow
\[ T_{xt} = \partial_t J_x + \partial_x J_t \]
\[ D_t = tT_{tt} - J_t , D_x = tT_{xt} - J_x \]

One can remove \( J_t \) by improvement
\[ T_{tt} \rightarrow \tilde{T}_{tt} = T_{tt} + \partial_t J_t \]
\[ T_{xt} \rightarrow \tilde{T}_{xt} = T_{xt} - \partial_x J_t \]

To show chiral conformal invariance, we need
\[ \partial_t J_x = 0 \] and it follows unitarity(?)
\[ \langle J_x(x, t)J_x(0) \rangle = f(x) , \rightarrow \langle \partial_t J_x(x, t) \partial_t J_x(0) \rangle = 0 \]
Reeh-Schlieder theorem ??

Symmetry determines \[ \langle \partial_t J_x(x, t) \partial_t J_x(0) \rangle = 0 \]
which means \[ \partial_t J_x |0\rangle = 0 \] from unitarity

**IF** analogue of Reeh-Schlieder theorem were true,
then we could conclude \[ \partial_t J_x = 0 \].

But with the given assumption alone, we **cannot** prove the theorem… (due to the lack of micro-causality in non-relativistic system)

Anyway, if this were overcome somehow, chiral conformal current would be conserved

\[ K_t = t^2 \tilde{T}_{tt} , \quad K_x = 0 \]
In 4-dimension...??

- No proof or counterexample (yet)
- All (renormalizable) classically scale inv actions are classically conf inv
- No scale but non-conf fixed point at 1-loop (2-loop?)
- Listen to the afternoon talk
- Like in 2d, there is a deep connection with a-theorem
Unexpected trace anomaly

• Dimensional analysis (4d):
\[
T_{\mu} = a(Euler) - c(Weyl^2) \\
+ bR^2 + b'\Box R + \epsilon \epsilon^{\rho\sigma\alpha\beta} R_{\rho\sigma\mu\nu} R^{\mu\nu}_{\alpha\beta} + \text{non anomalous terms}
\]

• Euler = \[ R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \] is important in “a-theorem” (“a” decreases along RG flow)

• Hirzebruch-Pontryagin term is CP violating but can appear in CFT (in principle)

• R^2 is inconsistent(?) for CFTs but it can appear in scale but non-CFT

• We’ll see these unexpected terms from freakology (non-orthodox holography)
a-theorem and $\varepsilon$ - conjecture

- Conformal anomaly $a$ in 4 dimension is monotonically decreasing along RG-flow.
- Komargodski and Schwimmer gave physical proof in flow between CFTs.
- The proof does not apply when the fixed points are scale invariant but not conformal invariant.
- Technically, it is problematic when they argue that dilaton (compensator) decouples from the IR sector. We cannot circumvent it without assuming “scale = conformal”.

- Looking forward to the complete proof in future.
Part 2. Holgraphic proof
Holographic claim

Scale invariant field configuration → Automatically invariant under the isometry of conformal transformation (AdS space)

Can be shown from Einstein eq + Null energy condition
Start from geometry

\[ d^{+1} \text{ metric with } d \text{ dim Poincare + } \]

\[ \text{scale invariance automatically selects } \text{AdS}_{d+1} \text{ space} \]

\[ ds^2 = \frac{dz^2}{z^2} + f(z)dx^2_a \]

\[ z \rightarrow \lambda z , \ t \rightarrow \lambda t , \ x \rightarrow \lambda x \]

\[ ds^2 = \frac{dz^2 + dx^2_a}{z^2} \]

\[ \delta x_a = 2(\epsilon^a x_b)x_a - (z^2 + x^b x_b)\epsilon_a , \ \delta z = 2(\epsilon^b x_b)z \]
Chiral case

2+1 metric with 2 dim chiral scale invariance automatically selects warped AdS\(_3\) space

The most general ansatz

\[ ds^2 = -a \frac{dt^2}{z^2} - 2b \frac{dt dx}{z} + c dx^2 + e \frac{dz^2}{z^2} + 2p \frac{dt dz}{z^2} + 2q \frac{dx dz}{z} \]

By coordinate change, it is warped AdS

\[ ds^2 = \frac{1}{c} \left( \frac{b dt}{z} - c dx \right)^2 + \frac{e (dz)^2 - (a + \frac{b^2}{c}) dt^2}{z^2} \]

Remarkably it has enhanced isometry

\[ \delta x = 2\epsilon \left( a + \frac{b^2}{c} \right) t z, \quad \delta t = \epsilon \left( a + \frac{b^2}{c} \right) t^2 + \epsilon e z^2 \]

\[ \delta x = \frac{2\epsilon b e}{c} \log z \]
Attention: space-time flipped Horava theory

Enhancement of “Isometry” requires d+1 diffeormorphism, so Horava theory which only preserves foliation preserving diff gives does not work.

\[ ds^2 = \frac{dz^2 + dx_a^2}{z^2} \]

\[ \delta x_a = 2(\epsilon^a x_b)x_a - (z^2 + x^b x_b)\epsilon_a, \quad \delta z = 2(\epsilon^b x_b)z \]

which is not foliation preserving diff

\[ \delta N = \partial_r (N f) \]
\[ \delta N^\mu = \partial_r (N^\mu f) + \partial_r \xi^\mu + \mathcal{L}_\xi N^\mu \]
\[ \delta g_{\mu\nu} = f \delta_r g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}. \]
Problem with Lorentz breaking

Non-trivial matter configuration may break AdS isometry

Example 1: non-trivial vector field

\[ A = A_\mu dx^\mu = \frac{adz}{z} \]

Example 2: non-trivial d-1 form field

\[ B = b \frac{dx_1 \cdots dx_d}{z^d} \]
But such a non-trivial configuration violates **Null Energy Condition**

Null energy condition: \( R_{\mu\nu} k^\mu k^\nu \geq 0 \), \( k^\mu k_\mu = 0 \)

(Ex)

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu + \lambda (A_\mu A^\mu)^2
\]

\[
R_{zz} + R_{tt} = (m^2 + 2\lambda a^2) a^2 = 0
\]

Basically, Null Energy Condition demands \( m^2 \) and \( \lambda \) are positive (= stability) and it shows \( a = 0 \)
More generically, strict null energy condition is sufficient to show scale = conformal from holography

Null energy condition: \[ R_{\mu\nu} k^\mu k^\nu \geq 0 \ , \ k^\mu k_\mu = 0 \]

strict null energy condition claims the equality holds if and only if the field configuration is trivial

• The trivial field configuration means that fields are invariant under the isometry group, which means that when the metric is AdS, the matter must be AdS isometric
On the assumptions

• Poincare invariance
  – Explicitly assumed in metric

• Discreteness of the spectrum
  – Number of fields in gravity are numerable

• Unitarity
  – Deeply related to null energy condition. E.g. null energy condition gives a sufficient condition on the area non-decreasing theorem of black holes.
On the assumptions: strict NEC

- In black hole holography
  - NEC is a sufficient condition to prove 
    area non-decreasing theorem for black hole horizon
  - Black hole entropy is monotonically increasing
- What does strict null energy condition mean?
  - Nothing non-trivial happens when the black hole entropy stays the same
- No information encoded in “zero-energy state”
- Holographic c-theorem is derived from the null energy condition
Freakolography and trace anomaly

- Consider space-time flipped Horava gravity whose dual is scale but non-conformal

\[ ds^2 = N^2 dr^2 + G_{\mu\nu} (dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr) \]

\[ S = \int N dr \sqrt{-G} d^d x (K^{\mu\nu} K_{\mu\nu} - \lambda K^2 + R + \Lambda) . \]

- Compute holographic trace anomaly

\[ \langle T^\mu_\mu \rangle = -2c \left( R_{\mu\nu} R^{\mu\nu} - \frac{\lambda}{4\lambda - 1} R^2 \right) \]

\[ = c \left( (\text{Euler} - \text{Weyl})^2 - \frac{2}{3} \frac{\lambda - 1}{4\lambda - 1} R^2 \right) . \]

- Cannot be conformal

- CP-odd term can appear with modification of he action with Levi-Civita tensor
What we leaned from holography

• **Full space-time diff** is tightly related to the emergence of conformal invariance

• It is possible to construct scale but non-conf geometry at the *sacrifice of full space-time diff* (spontaneous Lorentz symmetry breaking, Horava-like gravity…)

• Do they make sense? (violation of NEC, unitarity?…) Theorem in $d=2$?

• No holography? Or Freakoholography?
Summary

• Scale = Conformal invariance?

• Holography suggests the equivalence (but what happens in d>4?)

• Relation to c-theorem?

• Chiral scale vs conformal invariance

• Direct proof? Counterexample?