Scale vs Conformal invariance from holography

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Scale invariance
= Conformal invariance?

Scale = Conformal?

- QFTs and RG-groups are classified by scale invariant IR fixed point （Wilson's philosophy）

- Conformal invariance gave a (complete?) classification of 2D critical phenomena

- But scale invariance does not imply conformal invariance???

In equations…

- Scale invariance

$$x^\mu \to \lambda x^\mu$$

→Trace of energy-momentum (EM) tensor is a divergence of a so-called Virial current

$$T^\mu_{\ \mu} = \partial^\mu J_\mu \qquad D_\mu = x_\nu T_\mu^{\ \nu} - J_\mu$$

- Conformal invariance

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a^\mu x_\mu + a^2 x^2}$$

- EM tensor can be improved to be traceless

$$J_\mu = \partial^\nu L_{\mu\nu} \quad T^\mu_{\ \mu} \to \tilde{T}^\mu_{\ \mu} = 0$$

In today's talk

- I'll argue for the equivalence between scale and conformal from holographic viewpoint
- Importance of full diffeomorphism
- How to construct scale but non-conformal holographic models
- Surprising trace anomaly from holography

Part 1. From field theory

Maxwell theory in d > 4

- Scale invariance does NOT imply conformal invariance in d>4 dimension.

- (d>4) free Maxwell theory is an example
  (Jackiw and Pi, El-Showk, Nakayama and Rychkov)

- It is an isolated example because one cannot introduce non-trivial interaction

Maxwell theory in d > 4

- EM tensor and Virial current

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{g_{\mu\nu}}{4}(F_{\rho\sigma}F^{\rho\sigma})$$

$$T^{\mu}_{\ \mu} = \frac{4-d}{4}F_{\mu\nu}F^{\mu\nu} = \frac{4-d}{4}\partial_{\mu}(F^{\mu\rho}A_{\rho})$$

- Virial current $J^{\mu} = F^{\mu\rho}A_{\rho}$ is not a derivative so one cannot improve EM tensor to be traceless
- Dilatation current is not gauge invariant, but the charge is gauge invariant
- Does not satisfy Ward identity for conformal inv
- You may take a very specific gauge so that the gauge fixed action is conformal, but conformal symmetry does not commute with BRST

Zamolodchikov-Polchinski theorem (1988):

A scale invariant field theory is conformal invariant in (1+1) d when

1. It is unitary
2. It is Poincare invariant (causal)
3. It has a discrete spectrum

(4). Scale invariant current exists

(1+1) d proof

$$F(x^2) = z^4 \langle T(x)T(0) \rangle$$
$$G(x^2) = z^3 \bar{z} \langle \Theta(x)T(0) \rangle$$
$$H(x^2) = z^2 \bar{z}^2 \langle \Theta(x)\Theta(0) \rangle \; .$$

$$T \equiv T_{zz} \; , \quad \Theta \equiv T^{\mu}_{\mu}$$

$$\bar{\partial} T + 4 \partial \Theta = 0$$

According to Zamolodchikov, we define

$$C = 2(F - \frac{1}{2}G - \frac{3}{16}H)$$

$$\dot{C} \equiv x^2 \frac{d}{dx^2} C = -\frac{3}{4} H \leq 0$$ ← C-theorem!

At RG fixed point, $\dot{C} = 0$, which means $H = 0$

→ $\langle \Theta(x)\Theta(0) \rangle = 0 \iff \Theta = 0$

Reeh-Schlieder theorem

$$\langle \Theta(x)\Theta(0)\rangle = 0 \iff \Theta = 0$$

In CFT, this is true by state-operator correspondence. In generic QFT, it is due to the unitarity and Reeh-Schlieder theorem.

Unitarity tells you

$$\Theta(x)|0\rangle = 0$$

We then use Reeh-Schlieder theorem

$$O(x)|0\rangle = 0 \iff O(x) = 0$$

Proof is not that elementary: we need to use microcausality, so it is intrinsic to relativistic field theory (in Schrodinger theory, it is not true).

Hofman-Strominger theorem(?) (2011):
A chiral scale invariant field theory is chiral conformal invariant in (1+1) d when

1. It is unitary
2. It has a discrete spectrum

(3). Symmetry currents exist

But presumably (??) without Lorentz inv

$$H \ (L_1) : t \to t + \epsilon_t \quad D \ (L_0) : t \to \lambda t \ , \quad x \to x$$

$$P \ (\bar{L}_1) : x \to x + \epsilon_x$$

Then chiral conformal (and actually chiral Virasoro) appears

$$K \ (L_{-1}) \qquad t \to t + \epsilon t^2$$

Proof ??

H P + locality →

$$\partial_x T_{tx} + \partial_t T_{xx} = 0$$

$$\partial_x T_{tt} + \partial_t T_{xt} = 0$$

D + locality →

$$T_{xt} = \partial_t J_x + \partial_x J_t$$

$$D_t = tT_{tt} - J_t \ , D_x = tT_{xt} - J_x$$

One can remove J_t by improvement

$$T_{tt} \to \tilde{T}_{tt} = T_{tt} + \partial_t J_t$$

$$T_{xt} \to \tilde{T}_{xt} = T_{xt} - \partial_x J_t$$

To show chiral conformal invariance, we need

$\partial_t J_x = 0$ and it follows unitarity(?)

$$\langle J_x(x,t) J_x(0) \rangle = f(x) \ , \to \ \langle \partial_t J_x(x,t) \partial_t J_x(0) \rangle = 0$$

Reeh-Schlieder theorem ??

Symmetry determines $\langle \partial_t J_x(x,t) \partial_t J_x(0) \rangle = 0$

which means $\partial_t J_x |0\rangle = 0$ from unitarity

IF analogue of Reeh-Schlieder theorem were true, then we could conclude $\partial_t J_x = 0$.

But with the given assumption alone, we cannot prove the theorem… (due to the lack of micro-causality in non-relativistic system)

Anyway, if this were overcome somehow, chiral conformal current would be conserved

$$K_t = t^2 \tilde{T}_{tt} \ , \quad K_x = 0$$

In 4-dimension…??

- No proof or counterexample (yet)

- All (renormalizable) classically scale inv actions are classically conf inv
- No scale but non-conf fixed point at 1-loop (2-loop?)
- Listen to the afternoon talk

- Like in 2d, there is a deep connection with a-theorem

Unexpected trace anomaly

- Dimensional analysis (4d):

$$T^{\mu}_{\ \mu} = a(\mathrm{Euler}) - c(\mathrm{Weyl}^2)$$
$$+ bR^2 + b'\Box R + e\epsilon^{\rho\sigma\alpha\beta}R_{\rho\sigma\mu\nu}R^{\mu\nu}_{\ \ \alpha\beta} + \text{non anomalous terms}$$

- $\mathrm{Euler} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$ is important in "a-theorem" ("a" decreases along RG flow)
- Hirzebruch-Pontryagin term is CP violating but can appear in CFT (in principle)
- R^2 is inconsistent(?) for CFTs but it can appear in scale but non-CFT
- We'll see these unexpected terms from freakolography (non-orthodox holography)

a-theorem and ε - conjecture

- conformal anomaly a in 4 dimension is monotonically decreasing along RG-flow
- Komargodski and Schwimmer gave physical proof in flow between CFTs
- The proof does not apply when the fixed points are scale invariant but not conformal invariant
- Technically, it is problematic when they argue that dilaton (compensator) decouples from the IR sector. We cannot circumvent it without assuming “scale = conformal”

- Looking forward to the complete proof in future

Part 2. Holgraphic proof

Holographic claim

Scale invariant field configuration
→Automatically invariant under the isometry of conformal transformation (AdS space)

Can be shown from
Einstein eq + Null energy condition

Start from geometry

d+1 metric with d dim Poincare +
scale invariance automatically selects
AdS_{d+1} space

$$ds^2 = \frac{dz^2}{z^2} + f(z)dx_a^2$$

$$z \to \lambda z \ , \quad t \to \lambda t \ , \quad x \to \lambda x$$

$$ds^2 = \frac{dz^2 + dx_a^2}{z^2}$$

$$\delta x_a = 2(\epsilon^a x_b)x_a - (z^2 + x^b x_b)\epsilon_a \ , \delta z = 2(\epsilon^b x_b)z$$

Chiral case

2+1 metric with 2 dim chiral scale invariance automatically selects warped AdS_3 space

The most general ansatz

$$ds^2 = -a\frac{dt^2}{z^2} - 2b\frac{dtdx}{z} + cdx^2 + e\frac{dz^2}{z^2} + 2p\frac{dtdz}{z^2} + 2q\frac{dxdz}{z}$$

By coordinate change, it is warped AdS

$$ds^2 = \frac{1}{c}\left(\frac{bdt}{z} - cdx\right)^2 + \frac{e(dz)^2 - (a+\frac{b^2}{c})dt^2}{z^2}$$

Remarkably it has enhanced isometry

$$\delta z = 2\epsilon\left(a+\frac{b^2}{c}\right)tz \ , \quad \delta t = \epsilon\left(a+\frac{b^2}{c}\right)t^2 + \epsilon e z^2$$

$$\delta x = \frac{2\epsilon be}{c}\log z$$

Attention: space-time flipped Horava theory

Enhancement of “Isometry” requires d+1 diffeormorphism, so Horava theory which only preserves foliation preserving diff gives does not work.

$$ds^2 = \frac{dz^2 + dx_a^2}{z^2}$$

$$\delta x_a = 2(\epsilon^a x_b)x_a - (z^2 + x^b x_b)\epsilon_a \ , \delta z = 2(\epsilon^b x_b)z$$

which is not foliation preserving diff

$$\delta N = \partial_r(Nf)$$

$$\delta N^\mu = \partial_r(N^\mu f) + \partial_r \xi^\mu + \mathcal{L}_\xi N^\mu$$

$$\delta g_{\mu\nu} = f \delta_r g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} \ .$$

Problem with Lorentz breaking

Non-trivial matter configuration may break AdS isometry

Example 1: non-trivial vector field

$$A = A_\mu dx^\mu = \frac{a dz}{z}$$

Example 2: non-trivial d-1 form field

$$B = b\frac{dx_1 \cdots dx_d}{z^d}$$

But such a non-trivial configuration violates Null Energy Condition

Null energy condition: $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \ , \quad k^{\mu}k_{\mu} = 0$

(Ex)

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A_{\mu}A^{\mu} + \lambda(A_{\mu}A^{\mu})^2$$

$$R_{zz} + R_{tt} = (m^2 + 2\lambda a^2)a^2 = 0$$

Basically, Null Energy Condition demands m^2 and λ are positive (= stability) and it shows $a = 0$

More generically, strict null energy condition is sufficient to show scale = conformal from holography

Null energy condition: $R_{\mu\nu}k^\mu k^\nu \geq 0 \ , \quad k^\mu k_\mu = 0$

strict null energy condition claims the equality holds if and only if the field configuration is trivial

- The trivial field configuration means that fields are invariant under the isometry group, which means that when the metric is AdS, the matter must be AdS isometric

On the assumptions

- Poincare invariance
  - Explicitly assumed in metric
- Discreteness of the spectrum
  - Number of fields in gravity are numerable
- Unitarity
  - Deeply related to null energy condition. E.g. null energy condition gives a sufficient condition on the area non-decreasing theorem of black holes.

On the assumptions: strict NEC

- In black hole holography
  - NEC is a sufficient condition to prove area non-decreasing theorem for black hole horizon
  - Black hole entropy is monotonically increasing
- What does strict null energy condition mean?
  - Nothing non-trivial happens when the black hole entropy stays the same
- No information encoded in “zero-energy state”
- Holographic c-theorem is derived from the null energy condition

Freakolography and trace anomaly

- Consider space-time flipped Horava gravity whose dual is scale but non-conformal

$$\mathrm{ds}^2 = N^2 dr^2 + G_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr)$$

$$S = \int N dr \sqrt{-G} d^d x (K^{\mu\nu} K_{\mu\nu} - \lambda K^2 + R + \Lambda) \ .$$

- Compute holographic trace anomaly

$$\begin{aligned}\langle T^\mu_{\ \mu} \rangle &= -2c \left(R_{\mu\nu} R^{\mu\nu} - \frac{\lambda}{4\lambda - 1} R^2 \right) \\ &= c \left((\mathrm{Euler} - \mathrm{Weyl}^2) - \frac{2}{3} \frac{\lambda - 1}{4\lambda - 1} R^2 \right) \ .\end{aligned}$$

- Cannot be conformal
- CP-odd term can appear with modification of he action with Levi-Civita tensor

What we leaned from holography

- Full space-time diff is tightly related to the emergence of conformal invariance

- It is possible to construct scale but non-conf geometry at the sacrifice of full space-time diff (spontaneous Lonrentz symmetry breaking, Horava-like gravity…)
- Do they make sense? (violation of NEC, unitarity?…) Theorem in d=2?

- No holography? Or Freakoholography?

Summary

- Scale ＝ Conformal invariance？

- Holography suggests the equivalence (but what happens in d>4?)
- Relation to c-theorem?
- Chiral scale vs conformal invariance

- Direct proof ? Counterexample ?