Tutorial on Scale and Conformal symmetries in diverse dimensions

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These days mathematical physicists are closely investigating scale and conformal transformations. Familiarity with theories that are invariant against these transformations — at least on the classical, prequantized level — extends for over a hundred years. Nevertheless there remain features, not unknown to some, that generally have fallen into obscurity. Therefore I take this occasion to bring into light some of these forgotten topics.

I shall describe the relation between scale and conformal transformations, and state conditions that must be satisfied by invariant theories. Thereby exposing dimensional peculiarities and universalities of scale and conformal transformations

S-Y Pi & RJ, *J. Phys. A* **44**, 223001 (2011) [1101.4886];
... & S-H Ho, *J. Phys. A* **44**, 225401 (2011) [1101.4886]

Conformal Group of Transformations

(on a multi-component field Φ)

translations : $\delta_T^\sigma \Phi(x) = \partial^\sigma \Phi(x)$

Lorentz rotations : $\delta_L^{\sigma\tau} \Phi(x) = (x^\sigma \partial^\tau - x^\tau \partial^\sigma + \Sigma^{\sigma\tau})\ \Phi(x)$

dilation : $\delta_S \Phi(x) = (x^\tau \partial_\tau + d)\ \Phi(x)$

$d = \frac{D-2}{2}$ (free bosons in D dimensions)

special conformal (primary field) : $\delta_C^\sigma \Phi(x) = (2x^\sigma x^\tau - g^{\sigma\tau} x^2)\, \partial_\tau \Phi(x) + 2x_\tau \,(g^{\tau\sigma} d - \Sigma^{\tau\sigma})\, \Phi(x)$

$\Rightarrow$ $\delta x^\mu = -f^\mu(x)$

$f^\mu(x) = a^\mu, \quad \omega^{\mu\alpha} x_\alpha \quad (\omega^{\mu\alpha} = -\omega^{\alpha\mu}), \quad cx^\mu, \quad 2c_\alpha x^\alpha x^\mu - c^\mu x^2$

conformal Killing vector

$$\partial_\mu f_\nu + \partial_\nu f_\mu = \frac{2}{D}\, g_{\mu\nu}\, \partial_\alpha f^\alpha$$

Lorentz: $SO(D-1,1) \oplus$ Poincaré : $ISO(D-1,1)$,

$\oplus$Scale & Conformal : $SO(D,2)$

Relation between Scale and Conformal Symmetries

Conformal $\oplus$ Translation $\Rightarrow$ Lorentz $\oplus$ Dilation

$$[\delta_T^\sigma, \delta_C^\tau] = -2\, g^{\sigma\tau}\, \delta_S + 2\, \delta_L^{\sigma\tau}$$

Conditions for conformal symmetry:

1) Group theoretic: need scale symmetry

2) Dynamical: $\mathcal{L}\ (\partial_\mu \Phi, \Phi)$

$$V^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi}\ (g^{\mu\alpha}\, d - \Sigma^{\mu\alpha})\, \Phi$$

"Field virial" V^α must be total derivative: $V^\alpha = \partial_\beta\, \sigma^{\alpha\beta}$

$\Rightarrow$ energy momentum tenor $\theta^{\mu\nu}$ can be improved so that it is traceless

$$\theta^{\mu\nu} \to \theta^{\mu\nu}_{CCJ} \quad , \quad g_{\mu\nu}\, \theta^{\mu\nu}_{CCJ} = 0$$

$$\text{(Bessel-Hagen)} \quad J^\mu_f = \theta^{\mu\nu}_{CCJ}\, f_\nu \quad , \quad \partial_\mu J^\mu_f = 0$$

C Callan *et al. Ann. Phys.* **59**, 42 (1970).

NB: It is possible to have scale symmetry without conformal symmetry when field virial is not a total derivation

⇒ no traceless energy-momentum tensor exists, scale current is not of Bessel-Hagen form, involves terms beyond energy-momentum tensor

It happens that in many models scale symmetry is broken (*e.g.* by mass terms) but field virial is total derivative ⇒ obstacle to conformal invariance is scale non-invariance. "scale symmetry implies conformal symmetry" NOT generally true.

S Coleman & RJ, *Ann. Phys.* **67**, 552 (1971);
M Flato *et al. Ann. Phys.* **61**, 78 (1970)

Scale Symmetric but Conformally un-Symmetric Models

(A) $\mathcal{L}(\partial_\mu \varphi, \varphi) = \mathsf{L}\left(\frac{\partial_\mu \varphi \, \partial^\mu \varphi}{\varphi^{\frac{2D}{D-2}}}\right) \varphi^{\frac{2D}{D-2}}$

Scale invariant with any $L(z)$
Conformally invariant only with $L(z) = L_0 + L_1 z$

$$\Rightarrow \; \mathcal{L} = \frac{1}{2} \, \partial_\mu \varphi \, \partial^\mu \varphi - \lambda \, \varphi^{\frac{2D}{D-2}}$$

(B) Free Maxwell theory (vector potential based)

$$\mathcal{L} = -\frac{1}{4} \, F^{\alpha\beta} \, F_{\alpha\beta}$$

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$\theta^{\mu\nu} = -F^{\mu\alpha} \, F^\nu{}_\alpha + \frac{g^{\mu\nu}}{4} \, F^{\alpha\beta} \, F_{\alpha\beta}$$

$$\theta^\mu{}_\mu = (-1 + D/4) \; F^{\alpha\beta} \, F_{\alpha\beta} \qquad \neq 0 \;\; D \neq 4$$

(a) Scale Symmetry

$$\delta_S \, A_\alpha(x) = (x^\tau \, \partial_\tau + \frac{D-2}{2}) \; A_\alpha(x)$$

$$\delta_S F_{\alpha\beta}(x) = (x^\tau \, \partial_\tau + \frac{D}{2}) F_{\alpha\beta}(x)$$

$$J^\mu_S(x) = \theta^\mu{}_\alpha(x) \, x^\alpha + \underbrace{\frac{4-D}{2} \, F^{\mu\alpha}(x) \, A_\alpha(x)}_{\text{field virial}}$$

NB J_S^μ is gauge variant, but charge

$$\int d^{D-1}x \; J_S^0 \quad \text{is not}$$

(apart from surface)

(b) Conformal Symmetry

$$\begin{aligned}
\delta_C^\sigma \, A_\alpha(x) &= \text{primary field} \\
\delta_C^\sigma \, F_{\alpha\beta}(x) &= \text{not primary field} \\
&= \triangle^\sigma \, F_{\alpha\beta} + (D-4)(g_\alpha^\sigma \, A_\beta - g_\beta^\sigma \, A_\alpha) \\
& \quad \uparrow\!\underline{\quad\;} \;\; \text{primary field} \\
\delta_C^\sigma \, \mathcal{L} &= \partial_\mu \; \left[(2x^\sigma \, x^\mu - g^{\sigma\mu} \, x^2) \mathcal{L}) \right] + (4-D) \; F^{\sigma\tau} \, A_\tau \\
\text{(recall) } V^\alpha &= \frac{4-D}{2} \, F^{\alpha\beta} \, A_\beta \neq \partial_\beta \, \sigma^{\alpha\beta}
\end{aligned}$$

No conformal symmetry for $D \neq 4$

(based on vector potential)

Summary

$$\begin{aligned} J_f^\mu &= \theta^{\mu\alpha}\, f_\alpha + \frac{4-D}{2D}\,\partial_\alpha\, f^\alpha\, F^{\mu\beta}\, A_\beta \\ \partial_\mu\, J_f^\mu &= \frac{4-D}{2D}\,\partial_\mu\,\partial_\alpha\, f^\alpha\, F^{\mu\beta}\, A_\beta \\ \partial_\mu\,\partial_\alpha\, f^\alpha &= 0 \quad \text{except conformal} \end{aligned}$$

Final observation

$$\delta_f\, A_\alpha \;\neq\; \mathsf{L}_f\, A_\alpha \quad (D \neq 4)$$

↑ Lie derivative

$$\begin{aligned} \mathsf{L}_f\, A_\alpha &= f^\mu\,\partial_\mu\, A_\alpha + \partial_\alpha\, f^\mu\, A_\mu \\ &= f^\mu\, F_{\mu\alpha} + \partial_\alpha\,(f^\mu\, A_\mu) \\ \delta_f\, A_\alpha &= \mathsf{L}_f\, A_\alpha + \frac{D-4}{2D}\,\partial_\mu\, f^\mu\, A_\alpha \end{aligned}$$

Some Properties of Finite Conformal Transformations

$$x^\mu \to \bar{x}^\mu = (x^\mu + c^\mu x^2)/\sigma(x,c)$$

$$\sigma(x,c) \equiv 1 + 2c \cdot x \; + c^2 x^2$$

inversion: $x^\mu \to \bar{x}^\mu = x^\mu / x^2$

Conformal transformation: inversion | translation (c) | inversion

Small c: $\bar{x}^\mu = x^\mu + \underset{\llcorner\!\!\longrightarrow \text{ given before}}{\delta_c x^\mu}$

Large c:

$$\bar{x}^\mu = \underset{\substack{\uparrow \\ \text{translation}}}{\frac{c^\mu}{c^2}} + \underset{\substack{\uparrow \\ \text{dilation}}}{\frac{1}{c^2}} \; \underset{\substack{\uparrow \\ \text{Lorentz transformation}}}{\left(\delta^\mu_\nu - \frac{2\, c^\mu c_\nu}{c^2} \right)} \; \underset{\substack{\uparrow \\ \text{inversion}}}{\frac{x^\nu}{x^2}}$$

$$\Lambda^\mu_\nu(c) \equiv \delta^\mu_\nu - \frac{2 c^\mu c_\nu}{c^2}, \quad \text{det } \Lambda = -1$$

Lorentz improper ⇒ inversion improper

Finite Field Transformation $\Phi \to \bar{\Phi}$

Infinitesimal transformation:

$$\delta\Phi(x) \equiv \bar{\Phi}(x) - \Phi(x) = \left(2c \cdot x\, x^{\mu} - c^{\mu} x^{2}\right)\ \partial_{\mu}\Phi(x) + 2x \cdot c\, d\, \Phi(x)$$

$$- 2x_{\mu}\, c_{\nu}\, \Sigma^{\mu\nu}\, \Phi(x)$$

Spinless field Φ finite transformation:

$$\bar{\Phi}(\bar{x}) = [\sigma(x,c)]^{d}\ \Phi(x)$$

Vector field A_α finite transformation: spin complication

Consider $\Lambda_\alpha{}^\beta(x) = x^2 \, \frac{\partial}{\partial x^\alpha} \, \frac{x^\beta}{x^2} = \delta_\alpha{}^\beta - \frac{2x_\alpha \, x^\beta}{x^2}$

Define $A'_\alpha(x) = \Lambda_\alpha{}^\beta(x) \, A_\beta(x)$

Verify $\delta \, A'_\alpha(x) = \bar{A}'_\alpha(x) - A'_\alpha(x) =$

$\left(2c \cdot x \, x^\mu - c^\mu x^2\right) \; \partial_\mu \; A'_\alpha(x) + 2 \, x \cdot c \, d \; A'_\alpha(x)$

Spin term absent! A'_α transforms as a scalar!

$$\bar{A}'_\alpha(\bar{x}) = [\sigma(x,c)]^d \; A'_\beta(x)$$

$$\Rightarrow \bar{A}_\alpha(\bar{x}) = [\sigma(x,c)]^d \; \Lambda_\alpha{}^\beta(\bar{x}) \, \Lambda_\beta{}^\gamma(x) \, A_\gamma \, (x)$$

Lorentz transformation decoupled!

Dimensional Ladder for Scalar N-component field Φ

In D dimensions, N component field $\mathbf{\Phi}$

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \mathbf{\Phi} \cdot \partial^\mu \mathbf{\Phi} - \lambda\,(\mathbf{\Phi} \cdot \mathbf{\Phi})^{\frac{D}{D-2}}$$

is scale and conformally invariant

$$\begin{aligned} \delta_S\,\mathbf{\Phi}(x) &= \left(x^\tau\,\delta_\tau + \frac{D-2}{2}\right)\,\mathbf{\Phi}(x) \\ \delta_C^\sigma\,\mathbf{\Phi}(x) &= \left(2x^\sigma\,x^\tau - g^{\sigma\mu}\,x^2\right)\,\partial_\tau\,\mathbf{\Phi}(x) + (D-2)\,x^\sigma\,\mathbf{\Phi}(x) \end{aligned}$$

Poincaré: $ISO(D-1,1)$, Scale & Conformal: $SO(D,2)$

Continue to $D=1$ (time): $\mathbf{\Phi}(t,\mathbf{x}) \to \mathbf{q}(t)$

$$\mathcal{L} \to L = \frac{1}{2}\,\partial_t\,\mathbf{q} \cdot \partial_t\,\mathbf{q} - \lambda/q^2$$

Poincaré → time translation $\delta_t\,\mathbf{q} = \partial_t\,\mathbf{q}$

Scale $\to \delta_S\,\mathbf{q} = \left(t\,\partial_t - \frac{1}{2}\right)\,\mathbf{q}$

Conformal $\to \delta_C\,\mathbf{q} = \left(t^2\,\partial_t - t\right)\,\mathbf{q}$

Non relativistic conformal group $SO(D,2) \to SO(1,2)$

Dual to scalar field on AdS$_2$?

C Chamon *et al, Phys. Lett. B* **701**, 503 (2011) [1106.0726].