(2,0) theory on (singular) circle fibrations

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Superconformal symmetry

- The conformal algebra in $d$ (Minkowski) dimensions is $so(d,2)$.

- A superconformal algebra includes fermionic generators in a spinor representation.

- This is not possible for $d > 6$ (Nahm 1978).

- For $d = 6$, the bosonic subalgebra is $so(6,2) \oplus sp(2r)$ for some $r$, with fermionic generators in the $(8s, 2r)$ representation.

- Theories with $r = 2$ actually exist (Witten 1995)!
Six-dimensional (2,0) theory

- Completely classified by $ADE$-type with no further discrete or continuous parameters.
- Realized in ten-dimensional type $IIB$ string theory at codimension four singularity.
- $A$-series (and $D$-series) realized on coincident five-branes in eleven-dimensional $M$-theory (at an orientifold plane).
On a curved space-time

• Generally covariant theory on six-dimensional oriented spin manifold endowed with a conformal structure.

• But there is no generally covariant Lagrangian description.

• In fact, these theories do not even have any "fields" that obey classical differential "equations of motion"; they are intrinsically quantum theories.
Compactification on a circle

- Take one of the six space-time dimensions to be a circle of (small) radius $r$.

- The resulting low-energy effective theory in the remaining five dimensions then does admit a Lagrangian formulation.

- It is in fact maximally supersymmetric Yang-Mills theory in five dimensions with $ADE$-type gauge group and coupling constant $\sqrt{r}$. 
On a circle fibration

- A more general space-time might be a circle fibration with a conformal structure represented by a metric of the form

\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + (r(x))²(d\varphi + \theta_\mu(x)dx^\mu)^2 \]

- Here \( \varphi \) is the \( 2\pi \)-periodic fiber coordinate and \( x^\mu \), \( \mu = 0, 1, ..., 4 \) are coordinates for the remaining dimensions.

- The six-dimensional conformal structure is encoded in a five-dimensional metric \( g_{\mu\nu} \), a circle bundle connection \( \Theta_\mu \) and a scalar fiber radius function \( r \).

- An important role is played by the circle bundle curvature

\[ F_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu \]
How to find the reduced theory.

Outline:

• Consider the "abelian" version of (2,0) theory which does have a description in terms of fields and equations of motion in six dimensions.

• Reduce to five dimensions on a circle fibration.

• Find the corresponding Lagrangian description.

• Generalize this to the non-abelian theory.

• Check: Does supersymmetry work?

• Generalization: Example of a singular fibration.
Some references

• E. Witten,
  "Geometric Langlands from six dimensions",

• H. Linander and F. Ohlsson,
  "(2,0) theory on circle fibrations",
  JHEP 1201 (2012) 159,

• M. H., H. Linander and F. Ohlsson,
  work in progress.
The (2,0) tensor multiplet

This ”abelian” version of (2,0) theory does admit a description in terms of fields:

- A self-dual closed three-form $H$
  (a real $sp(4)$ singlet).

- Scalars $\Phi$
  (a real $sp(4)$ quintet).

- Chiral spinors $\Psi$
  (a symplectic Majorana $sp(4)$ quartet).
Conformally covariant equations of motion

- The self-duality condition $H = *H$ together with the Maxwell equation $dH = 0$.

- The massless Klein-Gordon equation $(D^M D_M + cR) \Phi = 0$ with conformal coupling $c = -1/5$.

- The massless Dirac equation $\Gamma^M D_M \Psi = 0$. 
The five-dimensional fields

Relate the six-dimensional fields $H$, $\Phi$ and $\Psi$ to five-dimensional fields $E$, $F$, $\Phi$ and $\Psi$:

$$H = E + F \wedge d\varphi$$

with a three-form $E$ and a two-form $F$ in five dimensions.

Rescale

$$\Phi = \frac{1}{r} \phi$$

$$\Psi = \frac{1}{r} \psi \otimes \eta$$

with a fixed spinor $\eta$ (to relate $SO(1,5)$ to $SO(1,4)$).
Reduced equations of motion

- For the tensor fields
  \[ E = \frac{1}{r} \star F + \theta \wedge F, \quad dE = 0, \quad dF = 0. \]

- For the scalar fields
  \[ D_\mu (r^{-1} D^\mu \phi) - r^{-1} m^2 \phi = 0 \quad \text{with} \]
  \[ m^2 = \frac{1}{5} R - \frac{1}{r^2} D_\mu r D^\mu r + \frac{3}{5r} D_\mu D^\mu r - \frac{1}{20} r^2 \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} = 0. \]

- For the spinor fields
  \[ \frac{2i}{r} \gamma^\mu D_\mu \psi - \frac{i}{r^2} \partial_\mu r \gamma^\mu \psi - \frac{1}{4} \mathcal{F}_{\mu\nu} \gamma^{\mu\nu} \psi = 0. \]
The abelian Lagrangian

These equations can be derived as the Euler-Lagrange equations of a five-dimensional covariant Lagrangian

\[ \mathcal{L} = \mathcal{L}_{\text{tensor}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{spinor}} \]

with

\[ \mathcal{L}_{\text{tensor}} = -\frac{1}{r} F \wedge * F + \theta \wedge F \wedge F \]

\[ \mathcal{L}_{\text{scalar}} = -\frac{1}{r} (D_\mu \phi D^\mu \phi + m^2 \phi^2) \]

\[ \mathcal{L}_{\text{spinor}} = \frac{i}{\sqrt{r}} \psi \gamma^\mu D_\mu \psi - \frac{1}{8} F_{\mu \nu} \psi \gamma^{\mu \nu} \psi \]
Minimal generalization to a non-abelian Lagrangian

- Promote $A$ to a connection on a principal $G$ bundle $E$ over the five-dimensional space-time and promote $\phi$ and $\psi$ to sections of $ad(E)$.

- The previous bilinear action terms are evaluated with the Killing form on Lie $G$.

- Add the familiar trilinear Yukawa couplings $\text{Tr}(\phi \psi \psi)$ and quartic potential terms $\text{Tr}([\phi,\phi]^2)$.
Does this give the right result?

The Lagrangian is strongly constrained by a symmetry originating in six-dimensional conformal invariance: A local conformal rescaling acts on the geometry as

\[
\begin{align*}
    g_{\mu\nu} & \rightarrow e^{-2\sigma} g_{\mu\nu} \\
    r & \rightarrow e^{-\sigma} r \\
    \theta_{\mu} & \rightarrow \theta_{\mu}
\end{align*}
\]

together with a transformation of the dynamical fields:

\[
\begin{align*}
    A_{\mu} & \rightarrow A_{\mu} \\
    \phi & \rightarrow e^{\sigma} \phi \\
    \psi & \rightarrow e^{\frac{3}{2}\sigma} \psi.
\end{align*}
\]

Our Lagrangian is the minimal possibility with this symmetry.
What about supersymmetry?

Supersymmetries are associated to conformal Killing spinors in six-dimensions:

\[
\left( D_M - \frac{1}{6} \Gamma_M \Gamma^N D_N \right) \epsilon = 0.
\]

(The number of linearly independent such chiral spinors varies between generically 0 and a maximum of 32 for conformally flat space taking R-symmetry into account.)

In five dimensions, this looks like

\[
\left( D_\mu - \frac{1}{2r} \partial_\nu r \gamma_\mu \gamma_\nu - \frac{ir}{8} F_{\rho\sigma} \gamma_\mu \gamma^{\rho\sigma} - \frac{ir}{4} F_{\mu\nu} \gamma_\nu \right) \epsilon = 0.
\]
The supersymmetry transformations schematically look like

\[
\begin{align*}
\delta A_\mu &= \psi \gamma_\mu \epsilon \\
\delta \phi &= \psi \epsilon \\
\delta \psi &= \left( F + D\phi + \frac{1}{r} \partial r \phi - \frac{F \phi}{r} + \phi^2 \right) \epsilon.
\end{align*}
\]

Invariance...

of the abelian equations of motion... ...is obvious,
of the abelian Lagrangian... ...is not too surprising,
of the non-abelian Lagrangian...

...is further strong evidence that this is indeed the correct reduction of ADE-type (2,0) theory on a circle fibration!
What about singular fibrations?

Sofar we have studied regular circle fibrations given by a free action of $S^1$ on a six-manifold $M$.

More generally, we may consider $S^1$ actions on $M$ with fixed points along a submanifold $W$. Clearly $W$ must be of even codimension in $M$.

codim $W = 2$ means that $M / S^1$ has a codimension one singularity (i.e. a boundary $B$).

codim $W = 4$ means that $M / S^1$ has a codimension three singularity (i.e. on a two-surface $\Sigma$).
The normal bundle action

The action of $e^{i\theta} \in S^1$ on $(T\Sigma) \cong \mathbb{C}^2$ is of the form $(z, w) \mapsto (e^{ip\theta} z, e^{iq\theta} w)$ for some (coprime) integers $p, q$ and has a fixed point at the origin.

An interesting example is $p=q=1$ for which

$$\mathbb{C}^2 / S^1 \cong \mathbb{R}^+ \times S^3 / S^1 \cong \mathbb{R}^+ \times S^2 \cong \mathbb{R}^3$$

by the Hopf fibration.

We should then consider Yang-Mills theory on the five-manifold $M / S^1$ with a coupling constant that goes to zero on the two-surface $\Sigma / S^1 \cong \Sigma$.

The curvature of the fibration is no longer closed but fulfils

$$d\mathcal{F} = 2\pi \delta_\Sigma$$
The gauge anomaly

The coupling
\[ \theta \wedge \text{Tr}(F \wedge F) \simeq F \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \]
leads to anomaly inflow on \( \Sigma \) with anomaly polynomial
\[ \text{Tr}(F \wedge F). \]

This must be cancelled by a chiral anomaly from additional degrees of freedom localized on \( \Sigma \).
The gauged WZW-model

A candidate for these degrees of freedom is chiral $G$ current algebra at level $k=1$ which has a Lagrangian description as a chirally gauged $G$ Wess-Zumino-Witten-model:

$$S = \frac{1}{8\pi} \int_{\Sigma} \text{Tr} \left( g^{-1} dg \wedge * g^{-1} dg \right)$$
$$+ \frac{1}{12\pi} \int_{B} \text{Tr} \left( g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right)$$
$$+ \frac{1}{4\pi} \int_{\Sigma} \text{Tr} \left( (A + * A) \wedge g^{-1} dg + gAg^{-1} \wedge * A \right).$$

(Here $B$ is an arbitrary three-manifold bounding $\Sigma$ and the degree of freedom $g$ is a $G$-valued map.)
Can this be seen more explicitly?

We return to the abelian case $G = U(1)$.

Away from the singularity locus $\Sigma$, the equations of motion for the tensor field read

$$-d\left(r^{-1} \ast F\right) + \mathcal{F} \wedge F = 0.$$

In view of the non-closedness of the fibration curvature $\mathcal{F}$, the extension over $\Sigma$ must involve a source term:

$$-d\left(r^{-1} \ast F\right) + \mathcal{F} \wedge F = \delta_\Sigma \wedge J,$$

where $J$ can be identified with the abelian chiral WZW-current $g^{-1} dg$ and fulfills

$$dJ = 2\pi F|_\Sigma.$$
A local model for the geometry

In the vicinity of $\Sigma$ we can take $M = R^{1,1} \times \text{(Taub-NUT)}$ with metric

$$ds^2 = 2d\sigma^+ d\sigma^- + Ud\mathbf{x} \cdot d\mathbf{x} + U^{-1}(d\varphi + \theta)^2$$

where

$$U = \frac{1}{|\mathbf{x}|} + \frac{1}{\lambda^2}, \quad \lambda = \text{constant}$$

$$d\theta = *dU$$

This geometry has $SO(1,1) \ltimes R^{1,1} \times SO(3)$ isometry, so solutions may be classified by two-dimensional momentum $(k^+, k^-)$ and three-dimensional spin $j$. 
Left-moving solutions with spherical symmetry

We make a left-moving, spherical symmetric Ansatz

\[ F = a(\sigma^+, |x|) \epsilon_{ijk} x^k \, dx^i \wedge dx^j + b(\sigma^+, |x|) \, x^i \, dx^i \wedge d\sigma^+ + c(\sigma^+, |x|) \, x^i \, dx^i \wedge d\sigma^- + d(\sigma^+, |x|) \, d\sigma^+ \wedge d\sigma^- \]

with arbitrary functions \(a, b, c\) and \(d\).

Imposing regularity at the origin \(|x| = 0\) gives the solutions

\[ F = \frac{f(\sigma^+)}{|x|(|x| + \lambda^2)^2} \, x^i \, dx^i \wedge d\sigma^+ \]

for which the world-sheet current is

\[ J = \frac{4\pi}{\lambda^2} f(\sigma^+) d\sigma^+ \]
Six-dimensional interpretation

From a six-dimensional perspective, these solutions are completely regular and take the form

\[ H = f(\sigma^+) d\sigma^+ \wedge d\Lambda. \]

Here we have introduced a (non-normalizable) one-form

\[ \Lambda = U^{-1}(d\phi + \theta) \]

the exterior derivative of which gives the famous normalizable self-dual two-form on Taub-NUT space (Pope 1978).
Some possible future projects

- How does supersymmetry act on the current $J$?
- Find the general solution in the abelian theory on Taub-NUT.
- Investigate the additional world-sheet degrees of freedom on more general singularities.
- What can be said about these in the non-abelian case?

To be continued...
Thank you aggies!