40 years of the Weyl anomaly

M. J. Duff

Physics Department
Imperial College London

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Abstract

- Classically, Weyl invariance

$$S(g, \phi) = S(g', \phi')$$

under

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x) \quad \phi' = \Omega(x)^\alpha \phi$$

implies

$$g^{\mu\nu} T_{\mu\nu} = 0$$

- But in the quantum theory

$$g^{\mu\nu} < T_{\mu\nu} > \neq 0$$

Over the period 1973-2012 this Weyl anomaly has found a variety of applications in quantum gravity, black hole physics, inflationary cosmology, string theory and statistical mechanics.

Recall flat space ancestry

- For spaces admitting conformal Killing vectors $\xi^i_\mu(x)$

$$\nabla_\mu \xi^i_\nu + \nabla_\nu \xi^i_\mu = \frac{2}{D} g_{\mu\nu} \nabla^\rho \xi^i_\rho$$

there is a classically conserved current

$$J^{i\nu} = \xi^i_\mu T^{\mu\nu}$$

- For example $SO(D,2)$ in flat Minkowski space
- But anomaly resides in the divergence of the dilatation current

$$\nabla_\nu < J^{i\nu} >= \frac{1}{D} \nabla^\rho \xi^i_\rho g^{\mu\nu} < T_{\mu\nu} > \neq 0$$

Coleman and Jackiw 1970

Timeline

- **1973**
  Discovery of the Weyl anomaly using dimensional regularization
  Capper and Duff

Timeline

- **1975**
  Supermultiplet of anomalies
  Ferrara and Zumino

Timeline

- **1976**
  Non-local effective lagrangian for trace anomalies
  Deser, Duff and Isham
  Zeta functions, heat kernels and anomalies
  Christensen
  Dowker
  Hawking

The heat kernel

- The one-loop effective action is given by

$$S_A = ln[det\Delta]^{-1/2}$$

where Δ is a conformally invariant d-dimensional operator. Its kernel $F(x, y, \rho)$ obeys the heat equation

$$\frac{\partial}{\partial\rho} F(x, y, \rho) + \Delta F(x, y, \rho) = 0$$

with the initial conditions

$$F(x, y, 0) = \delta(x, y)$$

The heat kernel

- One can express F as

$$F(x, y, \rho) = \sum_n e^{-\rho\Delta} \phi_n(x)\phi_n(y)$$

$$= \sum_n e^{-\rho\lambda_n} \phi_n(x)\phi_n(y)$$

where ϕ_n are the eigenfunctions of Δ with eigenvalues λ_n:

$$\Delta\phi_n = \lambda_n\phi_n$$

normalized according to

$$\int d^d x \sqrt{g}(x)\phi_n(x)\phi_m(x) = \delta_{mn}$$

b_4 coefficients

- The action may thus be written as

$$S_A = \int d\rho d^d x \rho^{-1} \sqrt{g}(x) A(x,\rho)$$

where $A(x,\rho) = F(x,x,\rho)$. $A(x,\rho)$ obeys an asymptotic expansion, valid for small ρ,

$$A(x,\rho) \sim \sum_n B_n(x) \rho^{n-\frac{d}{2}}$$

where

$$B_n = \int d^d x \sqrt{g} b_n(x) \qquad (1)$$

Zeta functions

- The Schwinger-DeWitt coefficients b_n are scalar polynomials, which are of order n in derivatives of the metric. In $d = 4$, for example, when Δ is the conformally invariant Laplacian acting on scalars:

$$\Delta = -\Box + \frac{1}{6}R$$

$$b_4 = \frac{1}{2880\pi^2}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + 30\Box R]$$

- Furthermore,

$$B_4 = n_0 + \zeta(0)$$

where n_0 is the number of zero modes and

$$\zeta(s) = \Sigma_n \;\; \lambda_n^{-s}$$

is defined only over the non-zero eigenvalues of Δ.

Timeline

- **1977**
  CFTs and the *a* and *c* coefficients
  Duff
  Trace anomalies and the Hawking effect
  Christensen and Fulling

CFTs

- Weyl anomalies appear in their most pristine form when CFTs are coupled to an external gravitational field. In this case
$$\mathcal{A} = g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{1}{(4\pi)^2}(cF - aG)$$
where F is the square of the Weyl tensor:
$$F = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$
G is proportional to the Euler density:
$$G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$
- Note no R^2 term.
- We ignore $\Box R$ terms whose coefficient can be adjusted to any value by adding the finite counterterm
$$\int d^4x\sqrt{g}R^2.$$

Central charges c and a

- In the CFT a and c are the central charges given in terms of the field content by

$$\bar{a} \equiv 720a = 2N_0 + 11N_{1/2} + 124N_1$$

$$\bar{c} \equiv 720c = 6N_0 + 18N_{1/2} + 72N_1$$

where N_s are the number of fields of spin s.

- In the notation of Duff 1977

$$(4\pi)^2 b = c \quad (4\pi)^2 b' = -a$$

Euler number

- When $F - G$ vanishes, anomaly reduces to

$$\mathcal{A} = A \frac{1}{32\pi^2} R^{*\mu\nu\rho\sigma} R^{*}{}_{\mu\nu\rho\sigma}$$

where

$$360A = \bar{c} - \bar{a} = 4N_0 + 7N_{1/2} - 52N_1$$

so that in Euclidean signature

$$\int d^4x \sqrt{g} g^{\mu\nu} T_{\mu\nu} = A\chi(M^4)$$

where $\chi(M^4)$ is the Euler number of spacetime.

Timeline

- **1978**
  Conformal (and axial) anomalies for arbitrary spin
  Christensen and Duff

Arbitrary spin

- Calculate b_4 for arbitrary (n, m) reps of Lorentz group, then physical anomaly given by

$$A = A(n, m) + A(n-1, m-1) - 2A(n-1/2, m-1/2)$$

so in total

$$A_{total} = 4N_0 + 7N_{1/2} - 52N_1 - 233N_{3/2} + 848N_2$$

where N_s are the number of fields of spin s.

- The b_4 coefficient for chiral reps (1/2,0) (1,0) etc also involve R*R unless we add (0,1/2) (0,1) etc

- **1980**
  Anomaly-driven inflation
  Starobinsky
  Vilenkin
  p-forms and inequivalent anomalies
  Duff and van Nieuwenhuizen
  Grisaru et al
  Siegel
  The path-integral approach to anomalies
  Fujikawa
  Bastianelli and van Nieuwenhuizin

Timeline

- **1981**
  Critical dimensions for bosonic and super strings
  Polyakov

Bosonic string

- In the first quantized theory of the bosonic string, one starts with a Euclidean functional integral

$$e^{-\Gamma} = \int \frac{D\gamma \, DX}{Vol(Diff)} \, e^{-S[\gamma, X]}$$

where

$$S[\gamma, X] = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$$

- As shown by Polyakov, the Weyl anomaly in the worldsheet stress tensor is given by

$$\gamma^{ij} < T_{ij} >= \frac{1}{24\pi}(D - 26)R(\gamma)$$

D is the contribution of the scalars while the -26 arises from the diffeomorphism ghosts that must be introduced into the functional integral.

Fermionic string

- In the case of the fermionic string, the result is

$$\gamma^{ij} < T_{ij} >= \frac{1}{16\pi}(D-10)R(\gamma)$$

- Thus the critical dimensions $D = 26$ and $D = 10$ correspond to the preservation of the two dimensional Weyl invariance $\gamma_{ij} \to \Omega^2(\xi)\gamma_{ij}$.

Timeline

- **1983**
  Conformal anomaly and W-Z consistency (no R^2)
  Bonora et al
  Anomaly in conformal supergravity
  Fradkin and Tseytlin

Timeline

- **1984**
  Local version of effective action
  Riegert

Local action

- Conformal operators

$$\sqrt{g}\Delta_d = \sqrt{g'}\Delta'_d$$

$$\Delta_2 = \Box$$

$$\Delta_4 \equiv \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}R\Box$$

Riegert

- Subsequent work by
  Antoniadis, Mazur and Mottola
- Local action

$$S_{anom} = \frac{b}{2}\int d^4x\sqrt{g}F\phi - \frac{b'}{2}\int d^4x\sqrt{g}[\phi\Delta_4\phi - (G - \frac{2}{3}\Box R)\phi]$$

Timeline

- **1985**
  Spacetime Einstein equations from vanishing worldsheet anomalies
  Callan et al

Timeline

- **1986**
  The *c*-theorem
  Zamolodchikov

Timeline

- **1988**
  c-theorem and/or *a*-theorem in four dimensions?
  Cardy
  Osborn
  Capelli et al
  Shore
  Shapere
  Antoniadis et al

Timeline

- **1993**
  Geometric classification of conformal anomalies in arbitrary dimensions
  Deser and Schwimmer

Timeline

- **1998**
  The holographic Weyl anomaly
  Henningson and Skenderis
  Graham and Witten
  Imbimbo et al
  Einstein manifolds and the *a* and *c* coefficients
  Gubser

Holography

- A distinguished coordinate system, boundary at $\rho = 0$

$$G_{MN} dx^M dx^N = \frac{{L_{d+1}}^2}{4} \rho^{-2} d\rho d\rho + \rho^{-1} g_{\mu\nu} dx^\mu dx^\nu$$

- The effective action may be written

$$S_B = \int d\rho d^d x \rho^{-1} \sqrt{g}(x) B(x, \rho)$$

where the specific form of $B(x, \rho)$ depends on initial action.

$$B(x, \rho) \sim \sum_n b_n(x) \rho^{n - \frac{d}{2}}$$

- Formal similarity with Schwinger-DeWitt coefficients, indeed $\mathcal{A} \sim b_4$ same for N=4 Yang-Mills but not in general.

Timeline

- **2000**
  Anomaly-driven inflation revived
  Hawking et al
  a and *c* and corrections to Newton's law
  Duff and Liu
  Anomalies and entropy bounds
  Nojiri et al

Corrections to Newton's law

- In his 1972 PhD thesis under Abdus Salam, the author calculated one-loop CFT corrections to Newton's law (Schwarzschild solution)

$$V(r) = \frac{G_4 M}{r}\left(1 + \frac{\alpha G_4}{r^2}\right),$$

where G_4 is the four-dimensional Newton's constant, $\hbar = c = 1$ and α is a purely numerical coefficient, soon recognized as the c coefficient in the Weyl anomaly

$$\alpha = \frac{8}{3\pi} c$$

N=4 Yang-Mills

- A particularly important example of a CFT is provided by $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$, for which

$$(N_1, N_{1/2}, N_0) = (N^2, 4N^2, 6N^2)$$

  Then

$$a = c = \frac{N^2}{4}$$

  and hence

$$\mathcal{A} = \frac{c}{(4\pi)^2}\left(2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2\right) = \frac{N^2}{32\pi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$$

- The contribution of a single $\mathcal{N} = 4$ $U(N)$ Yang-Mills CFT is

$$V(r) = \frac{G_4 M}{r}\left(1 + \frac{2N^2 G_4}{3\pi r^2}\right).$$

Randall-Sundrum

- Now fast-forward to 1999 when Randall and Sundrum proposed that our four-dimensional world is a 3-brane embedded in an infinite five-dimensional universe. They showed that there is an r^{-3} correction coming from the massive Kaluza-Klein modes
$$V(r) = \frac{G_4 M}{r}\left(1 + \frac{2{L_5}^2}{3r^2}\right).$$
where L_5 is the radius of AdS_5.
- Superficially, our 4D quantum correction seems unrelated to their 5D classical one.
- But through the miracle of AdS/CFT
$$N^2 = \frac{\pi L_5^3}{2G_5} \qquad G_4 = \frac{2G_5}{L_5}$$
the two are in fact equivalent. Duff and Liu

Timeline

- **2001**
  a and c and the graviton mass
  Dilkes et al
  Weyl cohomology revisited
  Mazur and Mottola

Timeline

- **2005**
  Anomalies as an infra-red diagnostic; IR free or interacting?
  Intriligator

Timeline

- **2006**
  Macroscopic effects of the quantum trace anomaly
  Mottola et al

Timeline

- **2007**
  Anomalies and the hierarchy problem
  Meissner

Timeline

- **2008**
  Viscosity bounds
  Buchel et al
  Conformal collider physics
  Hofman and Maldacena
  Weyl invariance and mass
  Waldron et al

Timeline

- **2009**
  Entanglement Entropy
  Nishioka
  Log corrections to black hole entropy
  Cai
  Solodukin
  Sen et al

Timeline

- **2010**
  Holographic c-theorems in arbitrary dimensions
  Myers et al
  Generalized mirror symmetry and trace anomalies
  Duff and Ferrara

Timeline

- **2011**
  Models for particle physics
  ’t Hooft
  Renormalization group and Weyl anomalies
  Percacci

M-theory on X^7

- We consider compactification of $(\mathcal{N} = 1, D = 11)$ supergravity on a 7-manifold X^7 with betti numbers $(b_0, b_1, b_2, b_3, b_3, b_2, b_1, b_0)$ and define a generalized mirror symmetry

$$(b_0, b_1, b_2, b_3) \to (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2)$$

under which

$$\rho(X^7) \equiv 7b_0 - 5b_1 + 3b_2 - b_3$$

changes sign

$$\rho \to -\rho$$

- The massless sectors of these compactifications have

$$f = 4(b_0 + b_1 + b_2 + b_3)$$

degrees of freedom.

- Generalized self-mirror theories are defined to be those for which $\rho = 0$

M-theory on X^7

- In backgrounds for which $F - G$ vanishes, the Weyl anomaly reduces to

$$T = A\frac{1}{32\pi^2}R^{*\,\mu\nu\rho\sigma}R^{*}{}_{\mu\nu\rho\sigma} \tag{2}$$

where

$$A = 2(c - a) \tag{3}$$

so that in Euclidean signature

$$\int d^4x\sqrt{g}T = A\chi(M^4) \tag{4}$$

where $\chi(M^4)$ is the Euler number of spacetime.

Anomalies

	Field	f	ΔA	$360A$	$360A'$	X^7
g_{MN}	$g_{\mu\nu}$	2	-3	848	-232	b_0
	$\mathcal{A}_\mu$	2	0	-52	-52	b_1
	$\mathcal{A}$	1	0	4	4	$-b_1 + b_3$
ψ_M	ψ_μ	2	1	-233	127	$b_0 + b_1$
	χ	2	0	7	7	$b_2 + b_3$
A_{MNP}	$A_{\mu\nu\rho}$	0	2	-720	0	b_0
	$A_{\mu\nu}$	1	-1	364	4	b_1
	A_μ	2	0	-52	-52	b_2
	A	1	0	4	4	b_3
total ΔA			0			
total A				$-\rho/24$		
total A'					$-\rho/24$	

Vanish without a trace!

- Remarkably, we find that the anomalous trace depends on ρ

$$A = -\frac{1}{24}\rho(X^7)$$

So the anomaly flips sign under generalized mirror symmetry and vanishes for generalized self-mirror theories. For $X^{(8-\mathcal{N})} \times T^{(\mathcal{N}-1)}$ with $\mathcal{N} \geq 3$ the anomaly vanishes identically.
Duff and Ferrara

- Equally remarkable is that we get the same answer for the total trace using the numbers of Grisaru et al.

Four curious supergravities

- Of particular interest are the four cases

$$(b_0, b_1, b_2, b_3) = (1, \mathcal{N} - 1, 3\mathcal{N} - 3, 4\mathcal{N} + 3)$$

  with $\mathcal{N} = 1, 2, 4, 8$, namely the four "curious" supergravities, discussed in Duff and Ferrara which enjoy some remarkable properties.
  $\mathcal{N} = 1$, 7 WZ multiplets, $f = 32$,
  $\mathcal{N} = 2$, 3 vector multiplets, 4 hypermultiplets, $f = 64$,
  $\mathcal{N} = 4$, 6 vector mutiplets, $f = 128$,
  $\mathcal{N} = 8$, $f = 256$.

O, H, C R theories

Field	360A	**O**	**H**	**C**	**R**
$g_{\mu\nu}$	848	1	1	1	1
B_μ	-52	7	6	0	0
S	4	28	16	10	7
ψ_μ	-233	8	4	2	1
χ	7	56	28	14	7
$A_{\mu\nu\rho}$	-720	1	1	1	1
$A_{\mu\nu}$	364	7	3	1	0
A_μ	-52	21	6	4	0
A	4	35	19	11	7
		$A = 0$	$A = 0$	$A = 0$	$A = 0$

Table: Vanishing anomaly in **O**, **H**, **C** **R** theories.

Fano plane

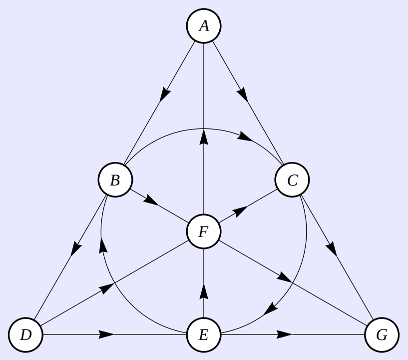

Figure: The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point. The truncation from 7 lines to 3 to 1 to 0 corresponds to the truncation from N=8 to N=4 to N=2 to N=1.

Type IIA

- In the case of $(\mathcal{N}=1, D=11)$ on $X^6 \times S^1$, or equivalently (Type IIA, D=10) on X^6,

$$A = -\frac{1}{24}\chi(X^6)$$

and so in Euclidean signature

$$\int d^4x\sqrt{g}g_{\mu\nu} < T^{\mu\nu} >= -\frac{1}{24}\chi(M^4)\chi(X^6) = -\frac{1}{24}\chi(M^{10})$$

where $\chi(M^4)$ is the Euler number of spacetime.

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