40 years of the Weyl anomaly

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Classically, Weyl invariance

\[ S(g, \phi) = S(g', \phi') \]

under

\[ g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x) \quad \phi' = \Omega(x)^\alpha \phi \]

implies

\[ g^{\mu\nu} T_{\mu\nu} = 0 \]

But in the quantum theory

\[ g'^{\mu\nu} < T_{\mu\nu} > \neq 0 \]

Over the period 1973-2012 this Weyl anomaly has found a variety of applications in quantum gravity, black hole physics, inflationary cosmology, string theory and statistical mechanics.
Recall flat space ancestry

- For spaces admitting conformal Killing vectors $\xi^i_\mu(x)$

  $$\nabla_\mu \xi^i_\nu + \nabla_\nu \xi^i_\mu = \frac{2}{D} g_{\mu\nu} \nabla^\rho \xi^i_\rho$$

  there is a classically conserved current

  $$J^{i\nu} = \xi^i_\mu T^{\mu\nu}$$

- For example $SO(D, 2)$ in flat Minkowski space

- But anomaly resides in the divergence of the dilatation current

  $$\nabla_\nu < J^{i\nu} > = \frac{1}{D} \nabla^\rho \xi^i_\rho g^{\mu\nu} < T_{\mu\nu} > \neq 0$$

Coleman and Jackiw 1970
1973
Discovery of the Weyl anomaly using dimensional regularization
Capper and Duff
1975
Supermultiplet of anomalies
Ferrara and Zumino
1976
Non-local effective lagrangian for trace anomalies
Deser, Duff and Isham
Zeta functions, heat kernels and anomalies
Christensen
Dowker
Hawking
The heat kernel

- The one-loop effective action is given by

\[ S_A = \ln[\det \Delta]^{-1/2} \]

where \( \Delta \) is a conformally invariant d-dimensional operator. Its kernel \( F(x, y, \rho) \) obeys the heat equation

\[ \frac{\partial}{\partial \rho} F(x, y, \rho) + \Delta F(x, y, \rho) = 0 \]

with the initial conditions

\[ F(x, y, 0) = \delta(x, y) \]
The heat kernel

One can express $F$ as

$$F(x, y, \rho) = \sum_n e^{-\rho \Delta} \phi_n(x) \phi_n(y)$$

$$= \sum_n e^{-\rho \lambda_n} \phi_n(x) \phi_n(y)$$

where $\phi_n$ are the eigenfunctions of $\Delta$ with eigenvalues $\lambda_n$:

$$\Delta \phi_n = \lambda_n \phi_n$$

normalized according to

$$\int d^d x \sqrt{g(x)} \phi_n(x) \phi_m(x) = \delta_{mn}$$
The action may thus be written as

$$S_A = \int d\rho d^d x \rho^{-1} \sqrt{g(x)} A(x, \rho)$$

where $A(x, \rho) = F(x, x, \rho)$. $A(x, \rho)$ obeys an asymptotic expansion, valid for small $\rho$,

$$A(x, \rho) \sim \sum_n B_n(x) \rho^{n - \frac{d}{2}}$$

where

$$B_n = \int d^d x \sqrt{g} b_n(x) \quad (1)$$
The Schwinger-DeWitt coefficients $b_n$ are scalar polynomials, which are of order $n$ in derivatives of the metric. In $d = 4$, for example, when $\Delta$ is the conformally invariant Laplacian acting on scalars:

$$\Delta = -\Box + \frac{1}{6}R$$

$$b_4 = \frac{1}{2880\pi^2}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + 30\Box R]$$

Furthermore,

$$B_4 = n_0 + \zeta(0)$$

where $n_0$ is the number of zero modes and

$$\zeta(s) = \sum_n \lambda_n^{-s}$$

is defined only over the non-zero eigenvalues of $\Delta$. 

### Zeta functions

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1977
CFTs and the $a$ and $c$ coefficients
Duff
Trace anomalies and the Hawking effect
Christensen and Fulling
Weyl anomalies appear in their most pristine form when CFTs are coupled to an external gravitational field. In this case

$$\mathcal{A} = g^{\mu \nu} \langle T_{\mu \nu} \rangle = \frac{1}{(4\pi)^2} (cF - aG)$$

where $F$ is the square of the Weyl tensor:

$$F = C_{\mu \nu \rho \sigma} C^{\mu \nu \rho \sigma} = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 2R_{\mu \nu} R^{\mu \nu} + \frac{1}{3} R^2,$$

$G$ is proportional to the Euler density:

$$G = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4R_{\mu \nu} R^{\mu \nu} + R^2,$$

Note no $R^2$ term.

We ignore $\Box R$ terms whose coefficient can be adjusted to any value by adding the finite counterterm

$$\int d^4x \sqrt{g} R^2.$$
Central charges $c$ and $a$

- In the CFT $a$ and $c$ are the central charges given in terms of the field content by

\[
\bar{a} \equiv 720a = 2N_0 + 11N_{1/2} + 124N_1
\]

\[
\bar{c} \equiv 720c = 6N_0 + 18N_{1/2} + 72N_1
\]

where $N_s$ are the number of fields of spin $s$.

- In the notation of Duff 1977

\[
(4\pi)^2 b = c \quad (4\pi)^2 b' = -a
\]
When $F - G$ vanishes, anomaly reduces to

$$A = \frac{1}{32\pi^2} R^{*\mu\nu\rho\sigma} R_{{\mu\nu\rho\sigma}}$$

where

$$360A = \bar{c} - \bar{a} = 4N_0 + 7N_{1/2} - 52N_1$$

so that in Euclidean signature

$$\int d^4x \sqrt{g} g^{\mu\nu} T_{\mu\nu} = A\chi(M^4)$$

where $\chi(M^4)$ is the Euler number of spacetime.
1978
Conformal (and axial) anomalies for arbitrary spin
Christensen and Duff
Arbitrary spin

- Calculate $b_4$ for arbitrary $(n, m)$ reps of Lorentz group, then physical anomaly given by

$$A = A(n, m) + A(n - 1, m - 1) - 2A(n - 1/2, m - 1/2)$$

so in total

$$A_{total} = 4N_0 + 7N_{1/2} - 52N_1 - 233N_{3/2} + 848N_2$$

where $N_s$ are the number of fields of spin $s$.

- The $b_4$ coefficient for chiral reps $(1/2,0)$ $(1,0)$ etc also involve $R^*R$ unless we add $(0,1/2) (0,1)$ etc
1980
Anomaly-driven inflation
Starobinsky
Vilenkin
$p$-forms and inequivalent anomalies
Duff and van Nieuwenhuizen
Grisaru et al
Siegel
The path-integral approach to anomalies
Fujikawa
Bastianelli and van Nieuwenhuizin
1981
Critical dimensions for bosonic and super strings
Polyakov
In the first quantized theory of the bosonic string, one starts with a Euclidean functional integral

\[ e^{-\Gamma} = \int \frac{D\gamma DX}{Vol(Diff)} e^{-S[\gamma, X]} \]

where

\[ S[\gamma, X] = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \]

As shown by Polyakov, the Weyl anomaly in the worldsheet stress tensor is given by

\[ \gamma^{ij} < T_{ij} >= \frac{1}{24\pi} (D - 26) R(\gamma) \]

D is the contribution of the scalars while the \(-26\) arises from the diffeomorphism ghosts that must be introduced into the functional integral.
In the case of the fermionic string, the result is

\[ \gamma^{ij} \langle T_{ij} \rangle = \frac{1}{16\pi} (D - 10) R(\gamma) \]

Thus the critical dimensions \( D = 26 \) and \( D = 10 \) correspond to the preservation of the two dimensional Weyl invariance \( \gamma_{ij} \rightarrow \Omega^2(\xi) \gamma_{ij} \).
1983
Conformal anomaly and W-Z consistency (no $R^2$)
Bonora et al
Anomaly in conformal supergravity
Fradkin and Tseytlin
1984
Local version of effective action
Riegert
Local action

- Conformal operators

\[ \sqrt{g} \Delta_d = \sqrt{g'} \Delta'_d \]

\[ \Delta_2 = \Box \]

\[ \Delta_4 \equiv \Box^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}R\Box \]

Rieger

- Subsequent work by Antoniadis, Mazur and Mottola

- Local action

\[ S_{anom} = \frac{b}{2} \int d^4x \sqrt{g} F \phi - \frac{b'}{2} \int d^4x \sqrt{g} [\phi \Delta_4 \phi - (G - \frac{2}{3} \Box R) \phi] \]
Timeline

- 1985
  Spacetime Einstein equations from vanishing worldsheet anomalies
  Callan et al
1986
The $c$-theorem
Zamolodchikov
1988

*c*-theorem and/or *a*-theorem in four dimensions?

Cardy
Osborn
Capelli et al
Shore
Shapere
Antoniadis et al
1993
Geometric classification of conformal anomalies in arbitrary dimensions
Deser and Schwimmer
1998
The holographic Weyl anomaly
Henningson and Skenderis
Graham and Witten
Imbimbo et al
Einstein manifolds and the $a$ and $c$ coefficients
Gubser
Holography

- A distinguished coordinate system, boundary at $\rho = 0$

$$G_{MN}dx^M dx^N = \frac{L_{d+1}^2}{4} \rho^{-2} d\rho d\rho + \rho^{-1} g_{\mu\nu} dx^\mu dx^\nu$$

- The effective action may be written

$$S_B = \int d\rho d^d x \rho^{-1} \sqrt{g(x)} B(x, \rho)$$

where the specific form of $B(x, \rho)$ depends on initial action.

$$B(x, \rho) \sim \sum_n b_n(x) \rho^{n-\frac{d}{2}}$$

- Formal similarity with Schwinger-DeWitt coefficients, indeed $A \sim b_4$ same for N=4 Yang-Mills but not in general.
Timeline

2000
Anomaly-driven inflation revived
Hawking et al

a and c and corrections to Newton’s law
Duff and Liu

Anomalies and entropy bounds
Nojiri et al
In his 1972 PhD thesis under Abdus Salam, the author calculated one-loop CFT corrections to Newton’s law (Schwarzschild solution)

\[ V(r) = \frac{G_4 M}{r} \left( 1 + \frac{\alpha G_4}{r^2} \right), \]

where \( G_4 \) is the four-dimensional Newton’s constant, \( \hbar = c = 1 \) and \( \alpha \) is a purely numerical coefficient, soon recognized as the \( c \) coefficient in the Weyl anomaly

\[ \alpha = \frac{8}{3\pi} c \]
A particularly important example of a CFT is provided by $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$, for which

$$(N_1, N_{1/2}, N_0) = (N^2, 4N^2, 6N^2)$$

Then

$$a = c = \frac{N^2}{4}$$

and hence

$$\mathcal{A} = \frac{c}{(4\pi)^2} \left( 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 \right) = \frac{N^2}{32\pi^2} \left( R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \right)$$

The contribution of a single $\mathcal{N} = 4$ $U(N)$ Yang-Mills CFT is

$$V(r) = \frac{G_4 M}{r} \left( 1 + \frac{2N^2G_4}{3\pi r^2} \right).$$
Now fast-forward to 1999 when Randall and Sundrum proposed that our four-dimensional world is a 3-brane embedded in an infinite five-dimensional universe. They showed that there is an $r^{-3}$ correction coming from the massive Kaluza-Klein modes

$$V(r) = \frac{G_4 M}{r} \left(1 + \frac{2L_5^2}{3r^2}\right).$$

where $L_5$ is the radius of AdS$_5$.

Superficially, our 4D quantum correction seems unrelated to their 5D classical one.

But through the miracle of AdS/CFT

$$N^2 = \frac{\pi L_5^3}{2G_5} \quad G_4 = \frac{2G_5}{L_5}$$

the two are in fact equivalent. Duff and Liu
2001

\( a \) and \( c \) and the graviton mass

Dilkes et al

Weyl cohomology revisited

Mazur and Mottola
2005
Anomalies as an infra-red diagnostic; IR free or interacting?
Intriligator
2006
Macroscopic effects of the quantum trace anomaly
Mottola et al
2007
Anomolies and the hierarchy problem
Meissner
2008
Viscosity bounds
Buchel et al
Conformal collider physics
Hofman and Maldacena
Weyl invariance and mass
Waldron et al
2009
Entanglement Entropy
Nishioka
Log corrections to black hole entropy
Cai
Solodukin
Sen et al
Timeline

2010
Holographic c-theorems in arbitrary dimensions
Myers et al
Generalized mirror symmetry and trace anomalies
Duff and Ferrara
Timeline

- **2011**
  - Models for particle physics
  - ’t Hooft
  - Renormalization group and Weyl anomalies
  - Percacci
We consider compactification of \((\mathcal{N} = 1, D = 11)\) supergravity on a 7-manifold \(X^7\) with betti numbers \((b_0, b_1, b_2, b_3, b_3, b_2, b_1, b_0)\) and define a generalized mirror symmetry

\[(b_0, b_1, b_2, b_3) \rightarrow (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2)\]

under which

\[\rho(X^7) \equiv 7b_0 - 5b_1 + 3b_2 - b_3\]

changes sign

\[\rho \rightarrow -\rho\]

The massless sectors of these compactifications have

\[f = 4(b_0 + b_1 + b_2 + b_3)\]

degrees of freedom.

Generalized self-mirror theories are defined to be those for which \(\rho = 0\).
In backgrounds for which $F - G$ vanishes, the Weyl anomaly reduces to

$$T = A \frac{1}{32\pi^2} R^*_{\mu\nu\rho\sigma} R^*_{\mu\nu\rho\sigma}$$  \hspace{1cm} (2)$$

where

$$A = 2(c - a)$$  \hspace{1cm} (3)$$

so that in Euclidean signature

$$\int d^4x \sqrt{g} T = A \chi(M^4)$$  \hspace{1cm} (4)$$

where $\chi(M^4)$ is the Euler number of spacetime.
## Anomalies

<table>
<thead>
<tr>
<th>Field</th>
<th>$f$</th>
<th>$\Delta A$</th>
<th>$360A$</th>
<th>$360A'$</th>
<th>$X^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{MN}$</td>
<td>$g_{\mu\nu}$</td>
<td>2</td>
<td>$-3$</td>
<td>848</td>
<td>$-232$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>2</td>
<td>0</td>
<td>$-52$</td>
<td>$-52$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>$-b_1 + b_3$</td>
</tr>
<tr>
<td>$\psi_M$</td>
<td>$\psi_\mu$</td>
<td>2</td>
<td>1</td>
<td>$-233$</td>
<td>127</td>
</tr>
<tr>
<td>$\chi$</td>
<td>2</td>
<td>0</td>
<td>7</td>
<td>7</td>
<td>$b_2 + b_3$</td>
</tr>
<tr>
<td>$A_{MNP}$</td>
<td>$A_{\mu\nu\rho}$</td>
<td>0</td>
<td>2</td>
<td>$-720$</td>
<td>0</td>
</tr>
<tr>
<td>$A_{\mu\nu}$</td>
<td>1</td>
<td>$-1$</td>
<td>364</td>
<td>4</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>2</td>
<td>0</td>
<td>$-52$</td>
<td>$-52$</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>$b_3$</td>
</tr>
</tbody>
</table>

| total $\Delta A$ | 0 |
| total $A$ | $-\rho/24$ |
| total $A'$ | $-\rho/24$ |
Remarkably, we find that the anomalous trace depends on $\rho$

$$A = -\frac{1}{24}\rho(X^7)$$

So the anomaly flips sign under generalized mirror symmetry and vanishes for generalized self-mirror theories. For $X^{(8-N)} \times T^{(N-1)}$ with $N \geq 3$ the anomaly vanishes identically.

Duff and Ferrara

Equally remarkable is that we get the same answer for the total trace using the numbers of Grisaru et al.
Four curious supergravities

Of particular interest are the four cases

\[(b_0, b_1, b_2, b_3) = (1, \mathcal{N} - 1, 3\mathcal{N} - 3, 4\mathcal{N} + 3)\]

with $\mathcal{N} = 1, 2, 4, 8$, namely the four “curious” supergravities, discussed in Duff and Ferrara which enjoy some remarkable properties.

$\mathcal{N} = 1$, 7 WZ multiplets, $f = 32$,
$\mathcal{N} = 2$, 3 vector multiplets, 4 hypermultiplets, $f = 64$,
$\mathcal{N} = 4$, 6 vector multiplets, $f = 128$,
$\mathcal{N} = 8$, $f = 256$. 
### O, H, C R theories

<table>
<thead>
<tr>
<th>Field</th>
<th>360A</th>
<th>O</th>
<th>H</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{\mu\nu}$</td>
<td>848</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$B_\mu$</td>
<td>−52</td>
<td>7</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S$</td>
<td>4</td>
<td>28</td>
<td>16</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>$\psi_\mu$</td>
<td>−233</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\chi$</td>
<td>7</td>
<td>56</td>
<td>28</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>$A_{\mu\nu\rho}$</td>
<td>−720</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A_{\mu\nu}$</td>
<td>364</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>−52</td>
<td>21</td>
<td>6</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$A$</td>
<td>4</td>
<td>35</td>
<td>19</td>
<td>11</td>
<td>7</td>
</tr>
</tbody>
</table>

$A = 0 \quad A = 0 \quad A = 0 \quad A = 0$

**Table:** Vanishing anomaly in O, H, C R theories.
Fano plane

Figure: The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point. The truncation from 7 lines to 3 to 1 to 0 corresponds to the truncation from N=8 to N=4 to N=2 to N=1.
In the case of \((\mathcal{N} = 1, D = 11)\) on \(X^6 \times S^1\), or equivalently (Type IIA, \(D=10\)) on \(X^6\),

\[
A = -\frac{1}{24} \chi(X^6)
\]

and so in Euclidean signature

\[
\int d^4x \sqrt{g} g_{\mu\nu} < T^{\mu\nu} >= -\frac{1}{24} \chi(M^4) \chi(X^6) = -\frac{1}{24} \chi(M^{10})
\]

where \(\chi(M^4)\) is the Euler number of spacetime.
Grateful to Stanley Deser, Emil Mottola and Marc Grisaru for discussions.

Thanks to the organizers for the invitation.