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- Restrict to N=2 SUSY, because
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  ... and N=4 is too simple.

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  … but complete classification is still not known
  … and tools for computing central charges are incomplete/conjectural.

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  ... and tools for computing central charges are incomplete/conjectural.

- The situation is much more confused for rank > 1 .

Outline

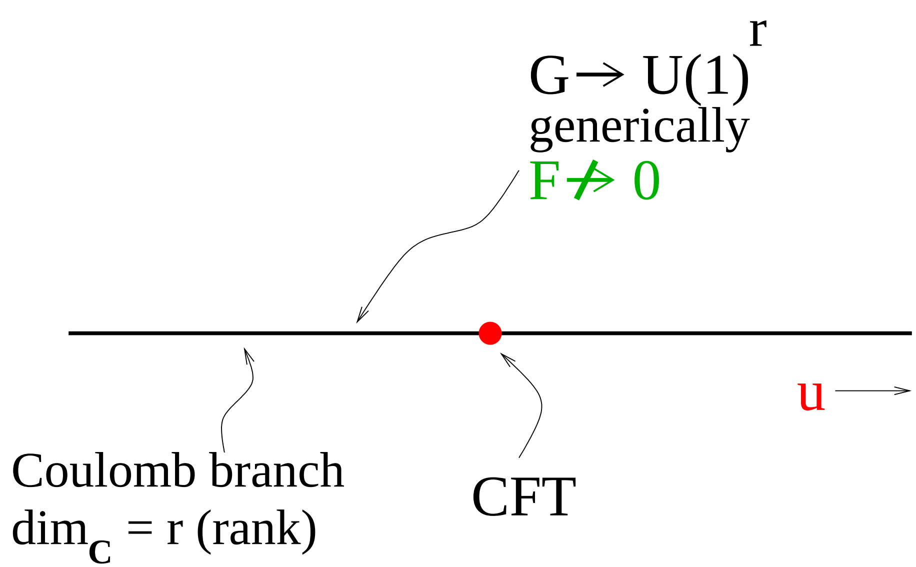
I	Vacuum structure of N=2 field theories
II	Rank 1 CFTs on the Coulomb branch
III	Mass deformations: at weak coupling
IV	Mass deformations: from S dualities
V	Mass deformations: from SW curves
VI	SW curves and CFTs: rank 1
VII	SW curves and CFTs: beyond rank 1

I Vacuum structure of N=2 field theories

• Lagrangian theories with gauge group G have a Coulomb branch of flat directions where $r = \text{rank}(G)$ complex scalars u of the vector multiplet get expectation values and $G \to U(1)^r$.

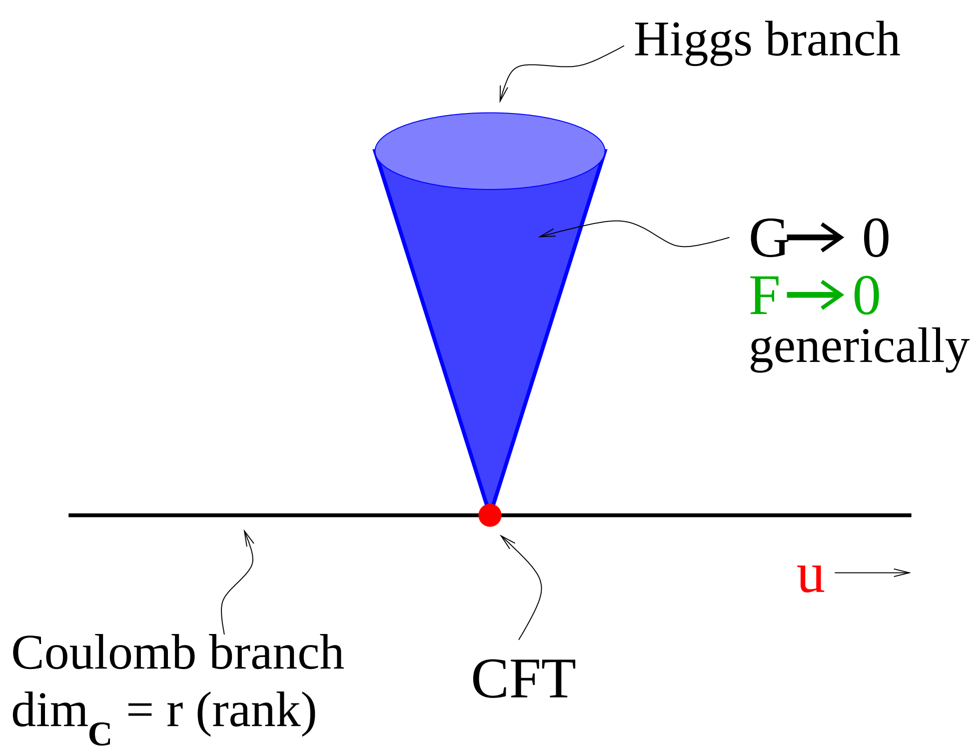
• $u \neq 0$ does not break the flavor symmetry F.

• In CFTs, $u \neq 0$ spontaneously breaks conformal invariance.



I Vacuum structure of N=2 field theories

- A CFT may also have a separate Higgs branch where complex scalars q of the hypermultiplets get expectation values..
- $q \neq 0$ generically completely Higgses G and breaks F.

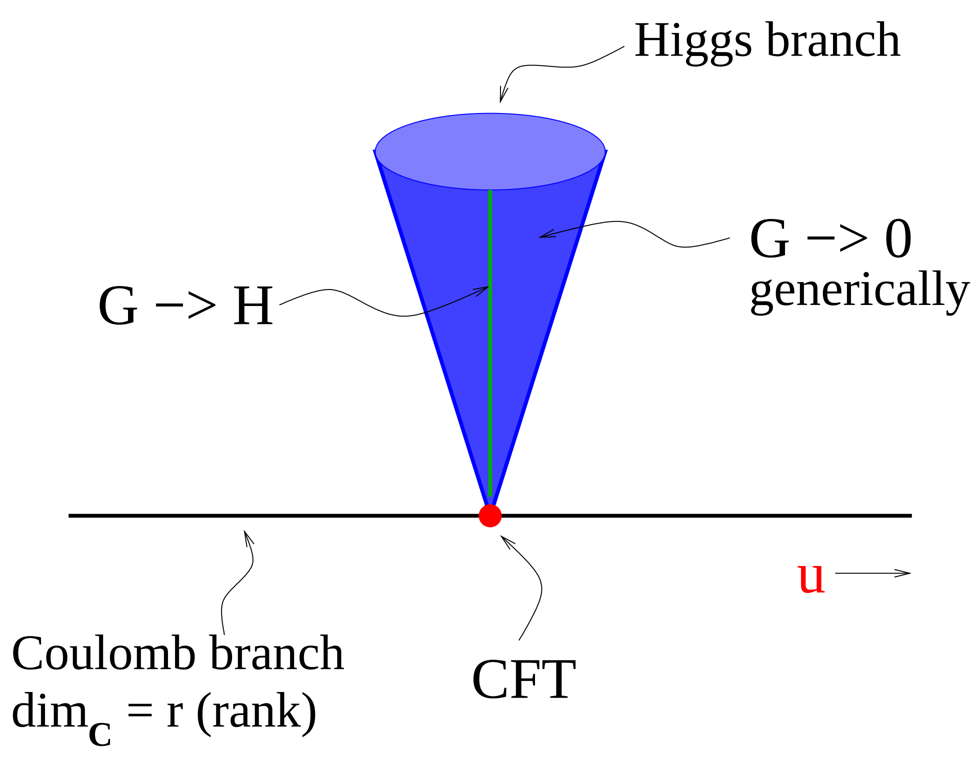


I Vacuum structure of N=2 field theories

• A CFT may also have a separate Higgs branch where complex scalars q of the half-hypermultiplets get expectation values..

• $q \neq 0$ generically completely Higgses G and breaks F.

• There can be submanifolds of the Higgs branch where the gauge group is not completely Higgsed.

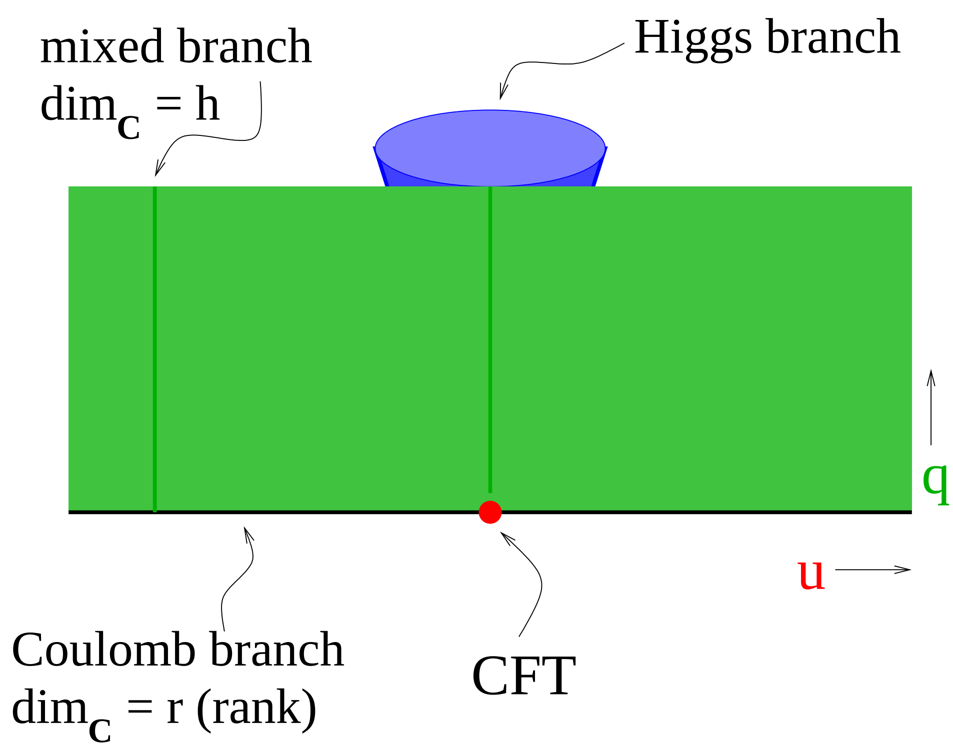


I Vacuum structure of N=2 field theories

• There can also be *mixed branches* where both h half-hypers, q, and $s \leq r$ vectors, u, get vevs.

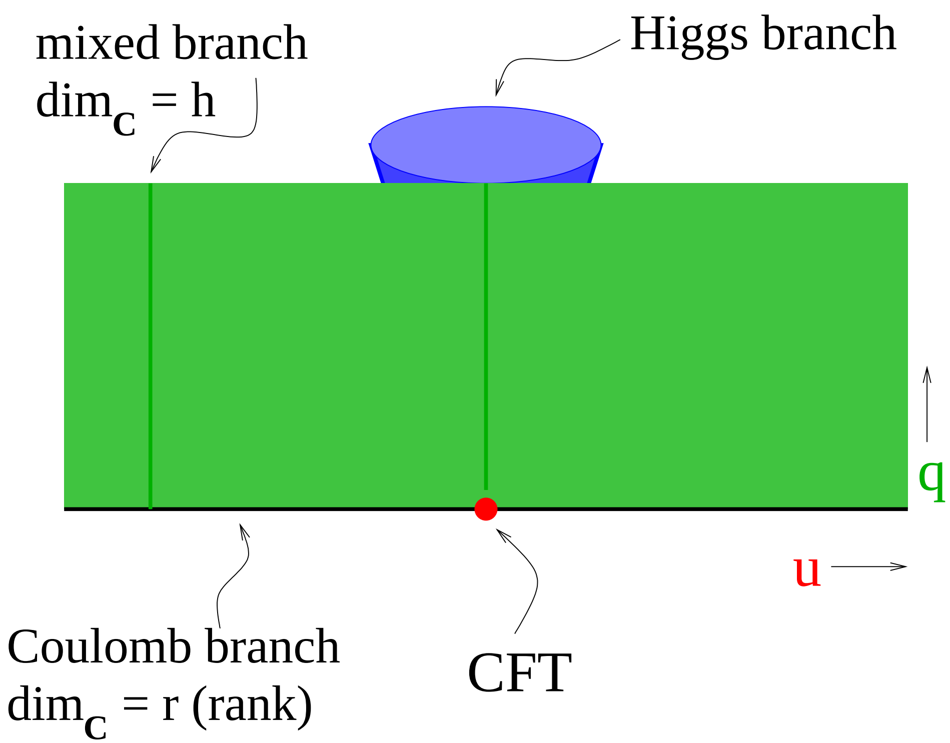
• $q \neq 0$ breaks F, but is not charged under the $U(1)^s$ gauge factors on the Coulomb branch.

• So mixed×Coulomb is smooth except at the CFT: at generic u the q are free neutral half-hypers in the low energy theory.



I Vacuum structure of N=2 field theories

- Later will be interested in *maximal mixed branches* where can turn on h half-hypers vevs, q, for *all* $r = \text{rank}(G)$ vectors, u.
- In this case there is a Higgs br. over every point of the Coulomb br.
- So maximal mixed $\times$ Coulomb = enlarged Coulomb branch.
- Occurs in any theory with hypers in *real* irrep, e.g., $SU(2) + 2 \cdot \mathbf{3}$ (*i.e.*, with adjoint hyper).



II Rank 1 CFTs on the Coulomb branch

• "Rank" here is the complex dimension of the Coulomb branch. It corresponds to the rank of the gauge group for Lagrangian theories.

• So rank 1 means that the low energy theory on the Coulomb branch is (generically) just $\mathfrak{u}(1)$ electrodynamics.

• A classification of the rank 1 N=2 CFTs is reviewed below; it is incomplete.

Massless Seiberg-Witten theory at rank 1 in a nutshell

• The low-energy physics on the Coulomb branch is encoded by a curve (complex torus) and a meromorphic 1-form λ_{SW} that vary holomorphically over the Coulomb branch.

• The complex structure of the torus described by the curve is the low energy $\mathfrak{u}(1)$ gauge coupling.

• The $\mathfrak{u}(1)$ charges (n_e, n_m) of a BPS state determine the homology class of a cycle on the torus, $\gamma = n_e[\alpha] + n_m[\beta]$, which determines the central charge (mass) of these states by Z $= \oint_\gamma \lambda_{SW}$.

II Rank 1 SCFTs on the Coulomb branch

• Write the curve in Weierstrass form

$$y^2 \;=\; x^3 \;+\; f(u)x \;+\; g(u)$$

• The 1-form λ_{SW} has zero residues and satisfies

$$\partial_u \lambda_{SW} \;=\; \frac{\mathrm{d}x}{y} \;+\; \partial_x(\star)\mathrm{d}x.$$

• u is a global complex coordinate on the Coulomb branch with scaling dimension $D(u) = d$.

• The curve is singular at the zeros of the discriminant

$$\Delta(u) \;\equiv\; 4 \cdot f^3 \;+\; 27 \cdot g^2 \;=\; 0.$$

These singularities in the u-plane physically correspond to a breakdown of the low-energy description since charged states are becoming massless at these points in moduli space.

• Scale invariance $\Rightarrow$ $\Delta(u) \sim u^n$, so the only singularity is at $u = 0$.

II Rank 1 SCFTs on the C.B.: Kodaira classification

name	curve	Δ	d	$\mathfrak{g}$
E_8	$y^2 = x^3 + 2u^5$	u^{10}	6	–
E_7	$y^2 = x^3 + u^3 x$	u^9	4	–
E_6	$y^2 = x^3 + u^4$	u^8	3	–
D_4	$y^2 = x^3 + 3\tau u^2 x + 2u^3$	u^6	2	$\mathfrak{su}(2)$
H_3	$y^2 = x^3 + u^2$	u^4	3/2	–
H_2	$y^2 = x^3 + ux$	u^3	4/3	–
H_1	$y^2 = x^3 + u$	u^2	6/5	–
$D_{n>4}$	$y^2 = x^3 + 3ux^2 + 4\Lambda^{-2(n-4)}u^{n-1}$	u^{n+2}	2	$\mathfrak{su}(2)$
$A_{n\geq 0}$	$y^2 = (x-1)(x^2 + \Lambda^{-(n+1)}u^{n+1})$	u^{n+1}	1	$\mathfrak{u}(1)$

• The possible scale-invariant singularities of rank-1 curves coincides with Kodaira's classification of degenerations of holomorphic families of elliptic curves over one variable [Minahan Nemeschansky hep-th/9608047, 9610076].

• The result is two infinite series and six "exceptional" curves.

• Λ is the UV strong coupling scale of the IR-free CFTs and τ is the marginal gauge coupling.

II Rank 1 SCFTs on the C.B.: Kodaira classification

name	curve	Δ	d	$\mathfrak{g}$
E_8	$y^2 = x^3 + 2u^5$	u^{10}	6	–
E_7	$y^2 = x^3 + u^3 x$	u^9	4	–
E_6	$y^2 = x^3 + u^4$	u^8	3	–
D_4	$y^2 = x^3 + 3\tau u^2 x + 2u^3$	u^6	2	$\mathfrak{su}(2)$
H_3	$y^2 = x^3 + u^2$	u^4	3/2	–
H_2	$y^2 = x^3 + ux$	u^3	4/3	–
H_1	$y^2 = x^3 + u$	u^2	6/5	–
$D_{n>4}$	$y^2 = x^3 + 3ux^2 + 4\Lambda^{-2(n-4)}u^{n-1}$	u^{n+2}	2	$\mathfrak{su}(2)$
$A_{n\geq 0}$	$y^2 = (x-1)(x^2 + \Lambda^{-(n+1)}u^{n+1})$	u^{n+1}	1	$\mathfrak{u}(1)$

- The $H_{1,2,3}$ were found as strongly-coupled fixed points in $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ AF theories in [PCA Douglas hep-th/9505062 & PCA Plesser Seiberg Witten hep-th/9511154].
- The $E_{6,7,8}$ strongly-coupled fixed point theories were constructed from strings by Ganor, Seiberg, and others in 1996, and as subsectors of lagrangian CFTs in [PCA Seiberg 0711.0054].

III Mass deformations: at weak coupling

• To get more refined information about these CFTs, ask:
What is their spectrum of relevant SUSY operators?

• From weakly coupled (lagrangian) theories:
The only N=2 relevant deformations are (complex) masses.
All masses transform in adjoint rep of flavor symmetry F.
So masses explicitly break $F \to U(1)^{\text{rank}(F)}$.

• Expect this pattern survives in non-Lagrangian theories:
If have flavor symmetry, can weakly gauge in N=2 invariant way
$\Rightarrow$ masses $\simeq$ adjoint Higgs vevs.
Mass operators in N=2 SCFT have $U(1)$ global currents in multiplet.

• Effect of masses on IR theory:
Preserve N=2 SUSY $\Rightarrow$ deformation of SW theory.
Complex deformation of SW curve, lessens singularity at $u = 0$.
Deforms SW 1 form consistent with RSK geometry.

III Mass deformations: at weak coupling (rank 1)

• **The $\mathbf{A_n}$ series** corresponds to a $\mathfrak{u}(1)$ gauge theory with massless charged hypermultiplets contributing $b = n + 1$ to the beta function.

• There are many ways of choosing charges to contributing a given amount to the beta function:

• E.g., n_a hypers of charge $\pm r_a$ give $b = \sum_a n_a r_a^2$, and a flavor symmetry $\mathfrak{f} = \oplus_a \mathfrak{u}(n_a)$.

• Similarly for **the $\mathbf{D_n}$ series** which are IR-free $\mathfrak{su}(2)$ theories for $n > 4$, and the conformal $\mathfrak{su}(2)$ theory for $n = 4$.

• The $\mathbf{D_4}$ **theory** has $b = 0$ with 4 fundamental hypers or with 1 adjoint hyper. The first has an $\mathfrak{so}(8)$ flavor symmetry, and the second an $\mathfrak{sp}(1)$ symmetry.

• The $\mathbf{D_{n>4}}$ **curves** are IR free with $b = 2(n - 4)$.

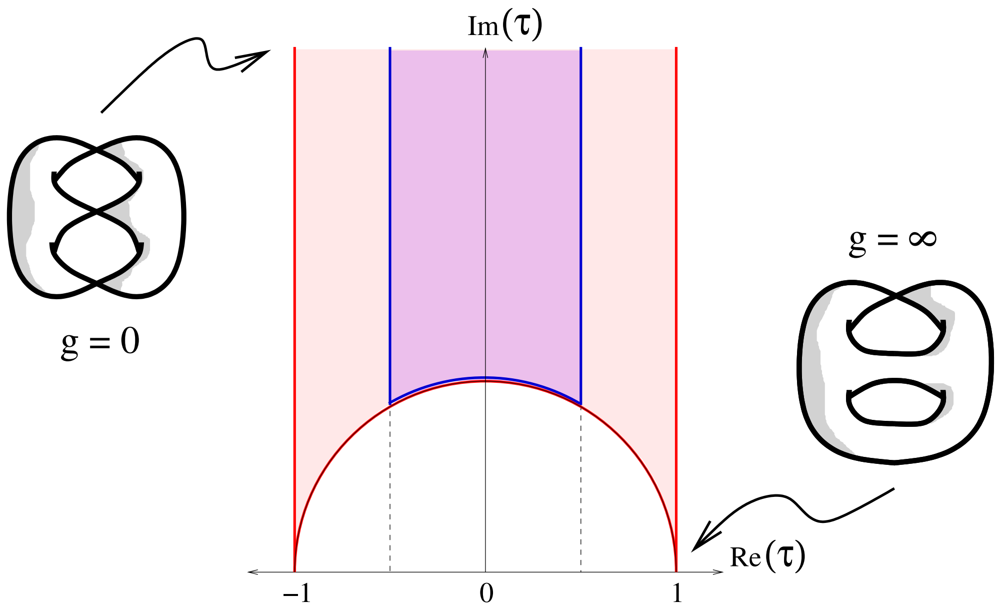
• Hypers can be in irreps of $\mathfrak{su}(2)$. Recall the $\mathbf{2r+1}$ are real and the $\mathbf{2s}$ are pseudoreal. N=2 SUSY implies only even numbers, $2n_r$, of each real irrep and any number m_s of the pseudoreal such that $\frac{1}{3}\Sigma_s m_s s(4s^2 - 1)$ is even (to avoid anomaly).

• These then contributes $b = \frac{4}{3}\Sigma_r n_r r(r+1)(2r+1) + \frac{1}{3}\Sigma_s m_s s(4s^2 - 1) - 8$ with a corresponding flavor symmetry $\mathfrak{f} = \oplus_r \mathfrak{sp}(n_r) \oplus_s \mathfrak{so}(m_s)$.

IV Mass deformations: from S dualities

- **Question:** What happens as $g \to \infty$ in N=2 gauge theories?

E.g., su(3) w/ $6 \cdot \mathbf{3}$ has S-duality group $\Gamma^0(2) \subset sl(2,\mathbb{Z})$ w/ fundamental domain in τ-plane containing $\mathrm{Im}(\tau) = 0$ ($g = \infty$):



IV Mass deformations: from S dualities

- **Answer:** [PCA Seiberg 0711.0054]
- The physics at $g = \infty$ of an N=2 lagrangian CFT with gauge algebra $\mathfrak{g}$, rank($\mathfrak{g}$) $= r$, is a weakly coupled scale-invariant gauge theory with gauge algebra $\widetilde{\mathfrak{g}}$ with smaller rank, rank($\widetilde{\mathfrak{g}}$) $= s < r$, which is coupled to an isolated rank-$(r-s)$ N=2 CFT:

	$\mathfrak{g}$	w/	hyper-plets	$\simeq$	$\widetilde{\mathfrak{g}}$	w/	(CFT	$\oplus$	hyper-plets)
coupling:	g		—		$1/g$		fixed		—
rank:	r		0		s		$r-s$		0

- Write the duality as

$$\mathfrak{g} \ \text{w/} \ \mathbf{r} \simeq \widetilde{\mathfrak{g}} \ \text{w/} \ (\widetilde{\mathbf{r}} \oplus \text{CFT}[d : \mathfrak{f}])$$

where $\mathbf{r}$ and $\widetilde{\mathbf{r}}$ are the hypers, and

- CFT[d : $\mathfrak{f}$] is a rank 1 CFT with mass dimension of the Coulomb branch moduli $D(u) = d$ and flavor symmetry algebra $\mathfrak{f}$.
- The coupling between $\widetilde{\mathfrak{g}}$ and the CFT is the standard gauge coupling: $\widetilde{\mathfrak{g}}$ gauges a subalgebra of $\mathfrak{f}$: the CFT provides "matter fields" charged under $\widetilde{\mathfrak{g}}$.

IV Mass deformations: from S dualities

- S-dualities imply relations among a, c (conformal central charges), k ($\mathfrak{f}$ central charge) for the $\mathfrak{g}$, $\tilde{\mathfrak{g}}$, and CFT factors.
- Along with matching flavor symmetries, $\mathbb{Z}_2$ anomaly, enables prediction of many low-rank examples.
- E.g., examples with 1 marginal operator [PCA Wittig 0712.2028]:

$\mathfrak{g}$	w/ $\mathbf{r}$	=	$\tilde{\mathfrak{g}}$	w/ $\tilde{\mathbf{r}}$	$\oplus$	CFT $[d:\mathfrak{f}]$
sp(3)	$\mathbf{14}\oplus 11\cdot\mathbf{6}$	=	sp(2)			$[6:E_8]$
su(6)	$\mathbf{20}\oplus\mathbf{15}\oplus\overline{\mathbf{15}}\oplus 5\cdot\mathbf{6}\oplus 5\cdot\overline{\mathbf{6}}$	=	su(5)	$\mathbf{5}\oplus\overline{\mathbf{5}}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}$		$[6:E_8]$
so(12)	$3\cdot\mathbf{32}\oplus\mathbf{32}'\oplus 4\cdot\mathbf{12}$	=	so(11)	$3\cdot\mathbf{32}$		$[6:E_8]$
G_2	$8\cdot\mathbf{7}$	=	su(2)	$\mathbf{2}$		$[6:\mathrm{sp}(5)]$
so(7)	$4\cdot\mathbf{8}\oplus 6\cdot\mathbf{7}$	=	sp(2)	$5\cdot\mathbf{4}$		$[6:\mathrm{sp}(5)]$
su(6)	$\mathbf{21}\oplus\overline{\mathbf{21}}\oplus\mathbf{20}\oplus\mathbf{6}\oplus\overline{\mathbf{6}}$	=	su(5)	$\mathbf{10}\oplus\overline{\mathbf{10}}$		$[6:\mathrm{sp}(5)]$
sp(2)	$12\cdot\mathbf{4}$	=	su(2)			$[4:E_7]$
su(4)	$2\cdot\mathbf{6}\oplus 6\cdot\mathbf{4}\oplus 6\cdot\overline{\mathbf{4}}$	=	su(3)	$2\cdot\mathbf{3}\oplus 2\cdot\overline{\mathbf{3}}$		$[4:E_7]$
so(7)	$6\cdot\mathbf{8}\oplus 4\cdot\mathbf{7}$	=	G_2	$4\cdot\mathbf{7}$		$[4:E_7]$
so(8)	$6\cdot\mathbf{8}\oplus 4\cdot\mathbf{8}'\oplus 2\cdot\mathbf{8}''$	=	so(7)	$6\cdot\mathbf{8}$		$[4:E_7]$
so(8)	$6\cdot\mathbf{8}\oplus 6\cdot\mathbf{8}'$	=	G_2			$[4:E_7]\oplus[4:E_7]$
sp(2)	$6\cdot\mathbf{5}$	=	su(2)			$[4:\mathrm{sp}(3)\oplus\mathrm{su}(2)]$
sp(2)	$4\cdot\mathbf{4}\oplus 4\cdot\mathbf{5}$	=	su(2)	$3\cdot\mathbf{2}$		$[4:\mathrm{sp}(3)\oplus\mathrm{su}(2)]$
su(4)	$\mathbf{10}\oplus\overline{\mathbf{10}}\oplus 2\cdot\mathbf{4}\oplus 2\cdot\overline{\mathbf{4}}$	=	su(3)	$\mathbf{3}\oplus\overline{\mathbf{3}}$		$[4:\mathrm{sp}(3)\oplus\mathrm{su}(2)]$
su(3)	$6\cdot\mathbf{3}\oplus 6\cdot\overline{\mathbf{3}}$	=	su(2)	$2\cdot\mathbf{2}$		$[3:E_6]$
su(4)	$4\cdot\mathbf{6}\oplus 4\cdot\mathbf{4}\oplus 4\cdot\overline{\mathbf{4}}$	=	sp(2)	$6\cdot\mathbf{4}$		$[3:E_6]$
su(3)	$\mathbf{3}\oplus\overline{\mathbf{3}}\oplus\mathbf{6}\oplus\overline{\mathbf{6}}$	=	su(2)	$n\cdot\mathbf{2}$		$[3:\mathfrak{f}]$

IV Mass deformations: from S dualities

- **Predictions for rank 1 CFTs**

d	$\mathfrak{f}$	$k_{\mathfrak{f}}$	$24 \cdot c$	$48 \cdot a$	$\mathbb{Z}_2$ anomaly?
6	$\mathfrak{e}_8$	12	124	190	no
6	$\mathfrak{sp}(5)$	7	98	164	yes
4	$\mathfrak{e}_7$	8	76	118	no
4	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$5 \oplus 8$	58	100	yes $\oplus$ no
3	$\mathfrak{e}_6$	6	52	82	no
3	$2 \leq \text{rank}(\mathfrak{f}) \leq 6$	$(8-n)/\mathrm{I}_{\mathfrak{su}(2) \hookrightarrow \mathfrak{f}}$	38-2n	68-2n	?

- At least one other entry — another mass deformation of the $d = 4$ Kodaira singularity — has been found by Chacaltana, Distler, and Tachikawa, using methods Jacques will probably talk about.

V Mass deformations: SW curves

- With masses the SW curve becomes

$$y^2 \;=\; x^3 \;+\; f(u, m_i)x \;+\; g(u, m_i) \qquad (1)$$

- 1-form λ_{SW} now allowed poles with constant residues m_i.
- The $U(1)^{\mathrm{rank}(F)}$ charges ("quark numbers") n_i of a BPS state enter the central charge by Z $= \oint_\gamma \lambda_{SW}$ with $\gamma = n_e[\alpha] + n_m[\beta] + n_i[\delta_i]$.
- Masses m_i transform as weights in the adjoint irrep of the global flavor symmetry algebra F.
- m_i appear in the curve (1) only in combinations invariant under of the Weyl group of F.
- m_i deform the discriminant by lower-order terms,

$$\Delta(u) = u^n + ... + P_{kd}(m_i)\, u^{n-k} + \cdots \,,$$

generically causing the singularity to split into up to n singularities on the u-plane.
- Since the order, n, of Δ is independent of $F \Rightarrow$ for $m_i \neq 0$ **different $F \Leftrightarrow$ different patterns of orders of zeros of Δ in u.**

V Mass deformations: SW curves

• The maximal complex (mass) deformations of Kodaira singularities:

$$y^2 = x^3 + x(M_2 u^3 + M_8 u^2 + M_{14} u + M_{20}) + (2u^5 + M_{12} u^3 + M_{18} u^2 + M_{24} u + M_{30}) \quad \mathfrak{e}_8$$
$$y^2 = x^3 + x(u^3 + M_8 u + M_{12}) + (M_2 u^4 + M_6 u^3 + M_{10} u^2 + M_{14} u + M_{18}) \quad \mathfrak{e}_7$$
$$y^2 = x^3 + x(M_2 u^2 + M_5 u + M_8) + (u^4 + M_6 u^2 + M_9 u + M_{12}) \quad \mathfrak{e}_6$$
$$y^2 = x^3 + x(3\tau u^2 + M_2 u + M_4) + (2u^3 + \widetilde{M}_4 u + M_6) \quad \mathfrak{so}(8)$$
$$y^2 = x^3 + x(M_{1/2} u + M_2) + (u^2 + M_3) \quad \mathfrak{u}(3)$$
$$y^2 = x^3 + x(u) + (M_{2/3} u + M_2) \quad \mathfrak{u}(2)$$
$$y^2 = x^3 + x(M_{4/5}) + (u) \quad \mathfrak{u}(1)$$
$$y^2 = x^3 + 3ux^2 + \Lambda^{-(n-4)} \widetilde{M}_n x + 4\Lambda^{-2(n-4)}(u^{n-1} + M_2 u^{n-2} + \cdots + M_{2n-2}) \quad \mathfrak{so}(2n)$$
$$y^2 = (x-1)(x^2 + \Lambda^{-(n+1)}[u^{n+1} + M_2 u^{n-1} + M_3 u^{n-2} + \cdots + M_{n+1}]) \quad \mathfrak{su}(n+1)$$

Dimensions of mass coefficients are the dimensions of the adjoint casimirs of the flavor groups on the right.

• For the A_n and D_n lagrangian theories, these flavor groups are only a particular case of all the possible flavor groups that we have seen can occur.

• E.g., the $\mathbf{D}_4$ **theory** is the conformal $\mathfrak{su}(2)$ which can have either 4 fundamental hypers or 1 adjoint hyper. The first has the $\mathfrak{so}(8)$ flavor symmetry shown above. The second $\mathfrak{sp}(1)$ symmetry, with curve with a sub-maximal mass deformation [Seiberg Witten 94]

$$y^2 = \prod_i (x - e_i u - e_i^2 M_2).$$

V Mass deformations: SW curves

• Seiberg and Witten worked out the 1-form for both mass deformations of the D_4 singularity.

• The 1-forms for the E_n maximal mass deformations were worked out in [Minahan Nemeschansky hep-th/9608047, 9610076].

• **General picture:**

strongly coupled CFTs	$\Leftrightarrow$	Lagrangian theories
singularity of curve	$\sim$	gauge algebra
N=2 complex deformations	$\sim$	matter representation

• **Program:** try to construct sub-maximal mass deformations of the exceptional non-Lagrangian singularities...

V Sub-maximal mass deformations

• To illustrate the problem, suppose we look for a sub-maximal mass deformation of the E_6 singularity with flavor symmetry $\mathfrak{f} = \mathfrak{su}(3)$.

• It then follows from an extension of [Shapere Tachikawa 0804.1957] that the discriminant will have only 4 zeros (with some multiplicities) for generic m_i (instead of the maximal 8), giving the following possible factorizations of the discriminant:

$$
\begin{aligned}
\Delta &\sim (u + ...)^5(u^3 + ...) \\
\Delta &\sim (u + ...)^4(u + ...)^2(u^2 + ...) \\
\Delta &\sim (u + ...)^3(u^2 + ...)^2(u + ...) \\
\Delta &\sim (u^2 + ...)^3(u^2 + ...) \\
\Delta &\sim (u^4 + ...)^2
\end{aligned}
$$

• A laborious computer search reveals 2 consistent factorization solutions...

V Sub-maximal mass deformations

• The 1st $\mathfrak{su}(3)$ solution is actually a 1-parameter (ν) family:

$$
\begin{aligned}
y^2 &= x^3 + 3N_2 x[u^2 + (1+\nu)N_2{}^3 + N_3{}^2] \\
&\qquad + [u^4 + u^2((1+2\nu)N_2{}^3 + 2N_3{}^2) \\
&\qquad\qquad + \nu(1+\nu)N_2{}^6 + (1+2\nu)N_2{}^3 N_3{}^2 + N_3{}^4] \\
\Delta &= -27[u^2 + (1+\nu)N_2{}^3 + N_3{}^2]^2[u^2 + (2+\nu)N_2{}^3 + N_3{}^2]^2
\end{aligned}
$$

• The 2nd $\mathfrak{su}(3)$ solution is

$$
y^2 = x^3 + u[3N_2 x(u - 4N_3) + u^3 - 12u^2 N_3 - u(N_2{}^3 - 48N_3{}^2) - 64N_3{}^3]
$$

$$
\Delta = -27u^2[u^3 - 12u^2 N_3 + u(N_2{}^3 + 48N_3{}^2) - 64N_3{}^3]^2
$$

• However, there *do not exist SW 1-forms for these curves*: these deformations are not compatible with $N = 2$ supersymmetry.

V Sub-maximal mass deformations: other examples

• Sub-maximal deformation of the E_7 singularity with flavor symmetry $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$ requires factoring a 9th-order polynomial into 6 generic zeros:

$$\Delta \sim (u + ...)^4 (u^5 + ...)$$
$$\Delta \sim (u + ...)^3 (u + ...)^2 (u^4 + ...)$$
$$\Delta \sim (u^3 + ...)^2 (u^3 + ...)$$

A systematic search reduces the problem to solving on the order of 800 polynomial relationships among 160 unknowns. Though highly over-constrained, it is computationally very difficult to determine whether there are any solutions.

• Sub-maximal deformation of the E_8 singularity with flavor symmetry $\mathfrak{sp}(5)$ requires factoring a 10th-order polynomial into 7 generic zeros:

$$\Delta \sim (u + ...)^4 (u^6 + ...)$$
$$\Delta \sim (u + ...)^3 (u + ...)^2 (u^5 + ...)$$
$$\Delta \sim (u^3 + ...)^2 (u^4 + ...)$$

A systematic search is much too difficult. Need some other method...

• [Chacaltana Distler 1106.5410] gives (strong constraints on) how masses will enter into these sub-maximal deformations.

V Sub-maximal mass deformations: isogenies

• An n-isogeny is an n-to-1 holomorphic map of a curve to itself that preserves the holomorphic 1-form. The existence of such a map implies that some of the zeros of the discriminant will have multiplicity n.

• There are three classical presentations of elliptic curves which are related by isogenies:

Legendre: $$y^2 = x^3 + f\,x + g$$

Jacobi: $$\eta^2 = \xi^4 + \alpha\,\xi^2 + \beta$$

Hessian: $$\gamma = \eta^3 + \delta\,\xi\eta + \xi^3$$

where f, g, α, β, γ, and δ are all functions of u and the m_i.

• The map between the Jacobi and Legendre forms is a 2-isogeny, while that between the Hessian and Legendre forms is a 3-isogeny.

V Sub-maximal mass deformations: isogenies

2-isogenies

- The H_2, D_4, and E_7 curves have 2-isogenous forms.

- **The $\mathbf{H_2}$ 2-isogenous curve** can only have a $\mathfrak{u}(1)$ flavor symmetry.

- **The $\mathbf{D_4}$ 2-isogenous curve** gives the $\mathfrak{sp}(1)$ (adjoint hyper) sub-maximal deformation of the D_4 singularity.

- **The $\mathbf{E_7}$ 2-isogenous curve** is

$$y^2 = x^3 + x\left(\beta - \frac{1}{3}\alpha^2\right) + \alpha\left(\frac{2}{27}\alpha^2 - \frac{1}{2}\beta\right).$$

with $\alpha = M_2 u + M_6$, $\beta = u^3 + M_8 u + M_{12}$.

∘ The mass parameter dimensions correspond to the dimensions of the Weyl invariants of an exceptional F_4 flavor symmetry.

∘ A SW 1-form for this curve exists, with masses transforming under the F_4 Weyl group.

V Sub-maximal mass deformations: isogenies

3-isogenies

- Only the H_3 and E_6 curves have a 3-isogenous forms.

- **The $\mathbf{H_3}$ 3-isogenous curve** can only have a $\mathfrak{u}(1)$ flavor symmetry.

- **The $\mathbf{E_6}$ 3-isogenous curve** is

$$y^2 = x^3 - x\,\delta\left(\gamma + \frac{1}{12}\delta^3\right) + \left(-\frac{1}{108}\delta^6 - \frac{1}{6}\gamma\delta^3 + \gamma^2\right)$$

with $\delta = M_2$, $\gamma = u^2 + M_6$.

∘ Mass dimensions are the dimensions of the Weyl invariants of exceptional G_2 flavor symmetry.

∘ A SW 1-form for this curve exists, with masses transforming under the G_2 Weyl group.

⋆ *It would be interesting to find any of these new curves by flowing to a fixed point in an AF theory or by S-duality from a "known" CFT.*

VI SW curves and CFTs: rank 1

Question: given the SW data, what can we learn about the CFT?

• By scaling to singularity, read off $d = \text{dim}(u)$

• Twisted SW theory relates [Shapere Tachikawa 0804.1957, PCA t.a.] the central charges ($k_{\mathfrak{f}}$, c, a) to (d, h, Z) where:

h is the number of neutral hypermultiplets at a *generic* point on the Coulomb branch (maximally-mixed branch)

Z is the number of singular points in the u-plane (counted *without* multiplicities) of the curve at *generic* masses.

• Find:

$$\begin{aligned} 48 \cdot a &= 12d + 2Zd - 2 + 2h \\ 24 \cdot c &= 2Zd + 4 + 2h \\ k_{\mathfrak{f}} &= 2d - h \end{aligned}$$

VI SW curves and CFTs: rank 1

• This comes from the topologically twisted path integral:

$$\int [\mathrm{d}u][\mathrm{d}q] A^{\chi} B^{\sigma} C^{n}\, \mathrm{e}^{-S_{\mathrm{SW}}}$$

• Here χ = Euler characteristic, σ = signature, n = instanton number of background space-time manifold and flavor bundle.

• $U(1)_R$ anomaly implies

$$\begin{aligned} \Delta R &= 2(2a-c)\cdot\chi + 3c\cdot\sigma - k_{\mathfrak{f}}\cdot n \\ &= R[A]\,\chi + R[B]\,\sigma + R[C]\,n + \tfrac{1}{2}(\chi+\sigma)\,r + (\tfrac{1}{4}\sigma + n)\,h \end{aligned}$$

• N=2 superconformal symmetry gives $R[\star] = 2D[\star]$.

• The tricky part is identifying general formulas for the measure factors in terms of the low energy data. For rank 1 these are, conjecturally,

$$\begin{aligned} A^2 &= \frac{du}{d\hat{a}} \qquad (\hat{a} \text{ is a "special coordinate" on the CB}) \\ B^8 &= \text{radical}[\Delta(u)] \\ C^{-1} &= u \end{aligned}$$

VI SW curves and CFTs: rank 1

• Adding these constraints to the S-duality predictions gives information about the number of zeros of $\Delta(u)$ (Z) and the number of neutral hypers (h) and how they transform under the flavor symmetry $\mathfrak{f}$:

d	$\mathfrak{f}$	Z	$2 \cdot$ h	$\mathfrak{f}$ irreps
6	$\mathfrak{e}_8$	10	0	-
6	$\mathfrak{sp}(5)$	7	10	$\mathbf{10}$
4	$\mathfrak{e}_7$	9	0	-
4	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	6	$(6,0)$	$\mathbf{6} \oplus \mathbf{1}$
3	$\mathfrak{e}_6$	8	0	-
3	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$	5	4	$(\mathbf{2},\mathbf{1})_+ \oplus (\mathbf{1},\mathbf{2})_-$
3	$\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$	5	4	$\mathbf{4}_0$
3	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	4	6	$(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \oplus 2 \cdot (\mathbf{1},\mathbf{1})$
3	$\mathfrak{sp}(2)$	4	6	$\mathbf{4} \oplus 2 \cdot \mathbf{1}$

• Which of the last four possibilities is realized is not yet known (though Yuji tells me it's either the first or the third).

• Explains the $\mathbb{Z}_2$-obstructions for the $\mathfrak{sp}(5)$ and $\mathfrak{sp}(3)$ factors as coming from the neutral hypermultiplet charged in a pseudoreal representation.

VII SW curves and CFTs: beyond rank 1

- The number of singular curves at rank 2 explodes (25 found in [PCA Wittig et al hep-th/0504070,0510226] and a bunch more in [Chacaltana Distler Tachikawa t.a.]). Can they be classified (à la Kodaira)?
- Given a singularity, is there an effective theory of its RSK deformations (i.e., N=2 mass deformations)?
- Given the mass deformations, can central charges (a, c, $k_{\mathfrak{f}}$) be determined from the SW data?
- More basically, can dim(Higgs) and flavor symmetry be determined?
- Even more basically, can the Coulomb branch scaling dimensions be uniquely determined? (cf. [Gaiotto Seiberg Tachikawa 1011.4568])
- Are there patterns? E.g., for lagrangian CFTs, though the pattern of mass deformations and their associated flavor algebras looks complicated, it follows from the much simpler algebraic structure of gauge algebras and matter representations. Is there a generalization of this algebraic structure for the non-lagrangian CFTs?